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A Lieb-Robinson Bound for the Toda System

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joint work with Robert Sims

Outline:

1. Motivation
2. Toda Lattice
3. Lieb-Robinson bounds

The Harmonic System

Let Λ be a finite subset of $\mathbb{Z}^d$. The harmonic Hamiltonian $H_h^\Lambda : \mathcal{X}_\Lambda \to \mathbb{R}$ is given by

$$H_h^\Lambda(\mathrm{x}) = \sum_{x\in\Lambda} p_x^2 + \omega^2 q_x^2 + \sum_{j=1}^{d} \lambda_j (q_x - q_{x+e_j})^2,$$

where $\mathrm{x} = (q_x, p_x)_{x\in\Lambda}$ and $\omega, \lambda_j \geq 0$.

For any integer $L \geq 1$ and each subset $\Lambda_L = (-L, L]^d \subset \mathbb{Z}^d$, the flow $\Phi_t^{h,L} : \mathcal{X}_{\Lambda_L} \to \mathcal{X}_{\Lambda_L}$ corresponding to $H_h^{\Lambda_L}$ may be explicitly computed.

A Lieb-Robinson Bound for the Harmonic System

The following is a Lieb-Robinson bound for the harmonic system (H. Raz, R. Sims [3]).

Theorem

Let X, Y be finite subsets of $\mathbb{Z}^d$ and take L_0 to be the minimal integer such that $X, Y \subset \Lambda_{L_0}$. For any $L \geq L_0$, denote by $\alpha_t^{h,L}$ the dynamics corresponding to $H_h^{\Lambda_L}$. For any $\mu > 0$ and any observables $A, B \in \mathcal{A}_{\Lambda_{L_0}}^{(1)}$ with supports in X and Y respectively, there exists positive numbers C and v_h, both independent of L, such that the bound

$$\left\| \{\alpha_t^{h,L}(A), B\} \right\|_{\infty} \leq C \left\| A \right\|_{1,\infty} \left\| B \right\|_{1,\infty} \min(|X|, |Y|) e^{-\mu(d(X,Y) - v_h|t|)}$$

holds for all $t \in \mathbb{R}$.

General Set-Up

We will consider the Toda system in $\mathbb{Z}$. To each integer $n \in \mathbb{Z}$, we associate an oscillator with position $q_n \in \mathbb{R}$ and momentum $p_n \in \mathbb{R}$. The state of the system is described by a sequence $\mathrm{x} = \{(q_n, p_n)\}_{n\in\mathbb{Z}}$, and the phase space is denoted by $\mathcal{X}$. The (infinite volume) Hamiltonian $H_T : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ for the Toda lattice is given by

$$H_T(\mathrm{x}) = \sum_{n\in\mathbb{Z}} \frac{p_n^2}{2} + V(q_{n+1} - q_n)$$

where $V(r) = e^{-r} + r - 1$ and $\mathrm{x} = \{(q_n, p_n)\}_{n\in\mathbb{Z}}$.

Hamilton's equations for this system are easy to write down: for each $n \in \mathbb{Z}$,

$$\dot{q}_n(t) = \frac{\partial H_T}{\partial p_n}(t) = p_n(t),$$

$$\begin{aligned}\dot{p}_n(t) = -\frac{\partial H_T}{\partial q_n}(t) &= V'(q_{n+1} - q_n) - V'(q_n - q_{n-1}) \\ &= e^{-(q_n(t) - q_{n-1}(t))} - e^{-(q_{n+1}(t) - q_n(t))}.\end{aligned}$$

Change of Variables

A convenient change of variables (commonly referred to as Flaschka variables [1], [2]) is: for each $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, set

$$a_n(t) = \frac{1}{2} e^{-(q_{n+1}(t) - q_n(t))/2} \text{ and } b_n(t) = -\frac{1}{2} p_n(t).$$

The corresponding system of equations of motion are

$$\begin{aligned} \dot{a}_n(t) &= a_n(t)\left(b_{n+1}(t) - b_n(t)\right) \\ \dot{b}_n(t) &= 2\left(a_n^2(t) - a_{n-1}^2(t)\right). \end{aligned} \tag{1}$$

We will consider the Toda Hamiltonian restricted to the Banach space $M = \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$. Each $\mathrm{x} \in M$ will be written as $\mathrm{x} = \{(a_n, b_n)\}_{n\in\mathbb{Z}}$. The norm on M is given by

$$\|\mathrm{x}\|_M = \max(\sup_n |a_n|, \sup_n |b_n|).$$

For the Toda system one can prove existence and uniqueness of the global solution on M. This is done in two stages. First one proves local existence of a solution and then extends it globally.

Local Existence

Theorem

If $x_0 = (a_0, b_0) \in M$ *then there exist* $\delta > 0$ *and a unique solution* $(a(t), b(t)) = \{(a_n(t), b_n(t))\}_{n\in\mathbb{Z}}$ *in* $C^\infty(I, M)$*, where* $I = (-\delta, \delta)$*, of the Toda equations (1) such that* $(a(0), b(0)) = (a_0, b_0)$*.*

Global Existence

Corresponding to each $\mathrm{x}_0 \in M$, define the following operators $H(t)$, $P(t) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $t \in I$, by setting

$$[H(t)f]_n = a_n(t)f_{n+1} + a_{n-1}(t)f_{n-1} + b_n(t)f_n,$$

$$[P(t)f]_n = a_n(t)f_{n+1} - a_{n-1}(t)f_{n-1}.$$

A short calculation shows that $P(t)$ and $H(t)$ are a Lax-Pair associated to (1), i.e.,

$$\frac{d}{dt}H(t) = [P(t), H(t)].$$

Since $P(t)$ is skew-symmetric, it generates a two-parameter family of unitary propagators $U(t,s)$ [4]. Moreover, the Lax equation implies that

$$H(t) = U(t,s)H(s)U(t,s)^* \quad \forall (t,s) \in I.$$

Hence $\|H(t)\|_2 = \|H(0)\|_2$ and therefore

$$\max\left(\|a(t)\|_\infty, \|b(t)\|_\infty\right) \leq \|H(t)\|_2 = \|H(0)\|_2,$$

implying that the solution can be globally extended.

Class of observables we consider

We will denote by $\mathcal{A}^{(1)}$ the set of all observables A for which

$$\|A\|_{1,\infty} = \sup_{n\in\mathbb{Z}} \max\left(\left\|\frac{\partial A}{\partial a_n}\right\|_\infty, \left\|\frac{\partial A}{\partial b_n}\right\|_\infty\right)$$

is finite. An observable A is said to be *supported* in $X \subset \mathbb{Z}$ if the observables $\frac{\partial A}{\partial a_n}$ and $\frac{\partial A}{\partial b_n}$ are identically zero for all $n \in \mathbb{Z} \setminus X$. The *support* of an observable A is the minimal set on which A is supported.

We will denote by α_t the Toda dynamics, i.e., $\alpha_t : \mathcal{A} \to \mathcal{A}$ defined by setting

$$\alpha_t(A) = A \circ \Phi(t),$$

where $\Phi(t)$ is the corresponding Toda flow.

If $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are functions of q_n's and p_n's, $n \in \mathbb{Z}$, then the Poisson bracket between them is defined as

$$\{A, B\}(\mathrm{x}) = \sum_{n \in \mathbb{Z}} \left(\frac{\partial A}{\partial q_n} \frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n} \frac{\partial B}{\partial q_n} \right),$$

where $\mathrm{x} = \{(q_n, p_n)\}_{n \in \mathbb{Z}}$. If $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are functions of a_n's and b_n's, $n \in \mathbb{Z}$, then the modified Poisson bracket is

$$\{A, B\}(\mathrm{x}) = \frac{1}{4} \sum_{n \in \mathbb{Z}} a_n \left(\frac{\partial A}{\partial a_n} \frac{\partial B}{\partial c_n} - \frac{\partial A}{\partial c_n} \frac{\partial B}{\partial a_n} \right),$$

where $\mathrm{x} = \{(a_n, b_n)\}_{n \in \mathbb{Z}}$ and $\frac{\partial}{\partial c_n} = \frac{\partial}{\partial b_{n+1}} - \frac{\partial}{\partial b_n}$.

Main Result

The following is a Lieb-Robinson bound for the Toda System.

Theorem

Let $\mathrm{x}_0 = \{(a_n, b_n)\}_{n\in\mathbb{Z}} \in M$. *Then, for every* $\mu > 0$ *there exists a number* $v = v(\mu, \mathrm{x}_0)$ *for which given any observables* $A, B \in \mathcal{A}^{(1)}$ *with finite supports* X *and* Y *respectively, the estimate*

$$|\{\alpha_t(A), B\}(\mathrm{x}_0)| \leq \|A\|_{1,\infty} \|B\|_{1,\infty} \sup_n |a_n| \sum_{n\in X, m\in Y} e^{-\mu(|n-m|-v|t|)}$$

holds for all $t \in \mathbb{R}$. *Here*

$$v(\mu, \mathrm{x}_0) = 18c\left(e^{\mu+1} + \frac{1}{\mu}\right),$$

where $c = c(\mathrm{x}_0) = \|H(0)\|_2$.

Sketch of the Proof A short calculation shows that

$$|\{\alpha_t(A), B\}(\mathrm{x}_0)| \leq \|A\|_{1,\infty} \|B\|_{1,\infty} \sup_n |a_n|$$

$$\times \sum_{n\in X, m\in Y} \frac{1}{2}\left(\left|\frac{\partial a_n(t)}{\partial a_m}\right| + \left|\frac{\partial b_n(t)}{\partial a_m}\right|\right)$$

$$+\frac{1}{4}\left(\left|\frac{\partial a_n(t)}{\partial b_{m+1}}\right| + \left|\frac{\partial a_n(t)}{\partial b_m}\right| + \left|\frac{\partial b_n(t)}{\partial b_{m+1}}\right| + \left|\frac{\partial b_n(t)}{\partial b_m}\right|\right).$$

Let $\Phi_n(t)$ denote the nth component of the flow $\Phi(t, \mathrm{x}_0)$, i.e.,

$$\Phi_n(t) = \begin{pmatrix} a_n(t) \\ b_n(t) \end{pmatrix}.$$

Clearly,

$$\Phi_n(t) = \Phi_n(0) + \int_0^t \begin{pmatrix} a_n(s)\,(b_{n+1}(s) - b_n(s)) \\ 2\left(a_n^2(s) - a_{n-1}^2(s)\right) \end{pmatrix} ds.$$

Let $\Phi_n'(t) = \frac{\partial \Phi_n(t)}{\partial z}$, where $z \in \{a_m, b_m, b_{m+1}\}$. WLOG we take $z = a_m$. Then differentiating the above equality with respect to z

$$\Phi_n'(t) = \delta_m(n) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{|e|\leq 1} \int_0^t D_{n+e}(s)\Phi_{n+e}'(s)ds,$$

where

$$D_{n+e}(s) = \begin{pmatrix} (b_{n+1}(s) - b_n(s))\,\delta_0(e) & a_n(s)(-\delta_0(e) + \delta_1(e)) \\ 4(a_n(s)\delta_0(e) - a_{n-1}(s)\delta_{-1}(e)) & 0 \end{pmatrix}.$$

For any $v = (x, y) \in \mathbb{R}^2$ take $\|v\| = \max(|x|, |y|)$. Then by taking the norm of both sides we get

$$\left\|\Phi_n'(t)\right\| \leq \delta_m(n) + c_1 \sum_{|e|\leq 1} \int_0^t \left\|\Phi_{n+e}'(s)\right\| ds,$$

where $c_1 = 6\,\|H(0)\|_2$.

By iterating the above inequality we obtain

$$\left\|\Phi_n'(t)\right\| \leq \sum_{k=|n-m|}^{\infty} \frac{(3c_1|t|)^k}{k!},$$

which implies that for any $\mu > 0$

$$\left\|\Phi_n'(t)\right\| \leq e^{-\mu(|n-m|-v|t|)}.$$

where $v = v(\mu, \mathrm{x}_0) = 18\,\|H(0)\|_2\,(e^{\mu+1} + \frac{1}{\mu})$.

Remark

Similar Lieb-Robinson bounds can be proven in general for the Toda Heirarchy.

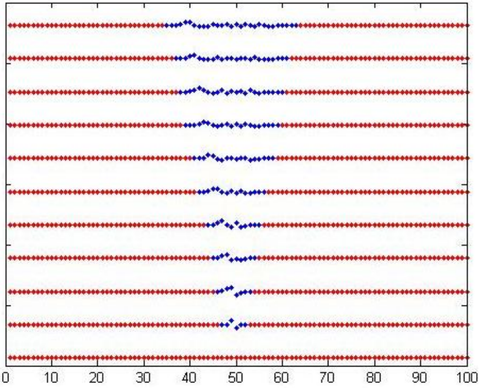
Similar Result for Another Class of Observables

Similar Lieb-Robinson bound is valid for another class of observables $\mathcal{A}^{(2)}$, consisting of all observables A and B for which

$$\|A\|_{2,\infty} = \max\left(\sqrt{\sum_n \left\|\frac{\partial A}{\partial a_n}\right\|_\infty^2}, \sqrt{\sum_n \left\|\frac{\partial A}{\partial b_n}\right\|_\infty^2}\right) < \infty.$$

Some Numerics

In this example we consider the Toda system in the finite volume and assume periodic boundary conditions. The number of particles $N = 100$. $a_n(0) = 1/2 \; \forall n$ and $b_n(0) = 0 \; \forall n \neq 50$, $b_{50}(0) = 1$. The values of $a_n(t)$, $1 \leq n \leq 100$, over the time interval $[0, 10]$ are plotted below.



Perturbations of the Toda system

Let $W:\mathbb{R}\to[0,\infty)$ such that $W\in C^2(\mathbb{R})$ and $W', W''\in L^\infty(\mathbb{R})$. We define a Hamiltonian $H^W:\mathcal{X}\to\mathbb{R}\cup\{\infty\}$ by setting

$$H_T^W(\mathrm{x}) = H_T(\mathrm{x}) + \sum_{n\in\mathbb{Z}} W(a_n),$$

where $\mathrm{x}=\{(a_n,b_n)\}_{n\in\mathbb{Z}}$. Hence the corresponding system of equations of motion are

$$\dot{a}_n(t) = a_n(t)\left(b_{n+1}(t)-b_n(t)\right),$$

$$\dot{b}_n(t) = 2\left(a_n^2(t)-a_{n-1}^2(t)\right)+R_n(t),$$

where

$$R_n(t)=\frac{1}{4}(W'(a_n(t))a_n(t)-W'(a_{n-1}(t))a_{n-1}(t)).$$

Local Existence

It is easy to see that there exists a unique local C^2 solution $(a(t), b(t))$ on $I(\mathrm{x}_0) = (-\delta, \delta)$ of the above Toda equations if the initial condition $\mathrm{x}_0 = (a_0, b_0) \in M$. Let α_t^W denote the perturbed Toda dynamics, i.e., $\alpha_t^W : \mathcal{A} \to \mathcal{A}$ defined by setting

$$\alpha_t^W(A) = A \circ \Phi_t^W,$$

where Φ_t^W is the corresponding perturbed Toda flow. Define the operators $P(t)$ and $H(t)$, $t \in I(\mathrm{x}_0)$ as before. Let $U(t,s)$ be the family of unitary propagators for $P(t)$. A simple calculation shows that

$$\frac{d}{dt} H(t) = [P(t), H(t)] + R(t),$$

where $R(t)$ is a bounded linear operator in $\ell^2(\mathbb{Z})$ given by

$$[R(t)f]_n = R_n(t) f_n.$$

Let $\tilde{H}(t) = U(t,s)^* H(t) U(t,s)$. Then $\|\tilde{H}(t)\|_2 = \|H(t)\|_2$ and by the similar earlier calculations we get

$$\frac{d}{dt}\tilde{H}(t) = U(t,s)^* R(t) U(t,s).$$

Hence

$$\tilde{H}(t) = \tilde{H}(0) + \int_0^t U(\tau,s)^* R(\tau) U(\tau,s) d\tau.$$

By taking the norm we get

$$\|H(t)\|_2 \leq \|H(0)\|_2 + \frac{1}{2}\left\|W'\right\|_\infty \int_0^t \|H(\tau)\|_2 \, d\tau.$$

By Gronwall's lemma he have

$$\|H(t)\|_2 \leq c_1 e^{c_2|t|},$$

where $c_1 = \|H(0)\|_2$ and $c_2 = \frac{1}{2}\left\|W'\right\|_\infty$, implying the global existence of the solution.

Another Main Result

Theorem

Let $\mathrm{x}_0 = \{(a_n, b_n)\}_{n\in\mathbb{Z}} \in M$. *Then, for every* $\mu > 0$ *and* $T > 0$ *there exists a number* $v = v(\mu, \mathrm{x}_0, W, T)$ *for which given any observables* $A, B \in \mathcal{A}^{(1)}$ *with finite supports* X *and* Y *respectively, the estimate*

$$|\{\alpha_t^W(A), B\}(\mathrm{x}_0)| \leq \|A\|_{1,\infty} \|B\|_{1,\infty} \sup_n |a_n| \sum_{n\in X, m\in Y} e^{-\mu(|n-m|-v|t|)}$$

holds for all $t \in (-T, T)$. *Here*

$$v(\mu, \mathrm{x}_0, W, T) = c\left(e^{\mu+1} + \frac{1}{\mu}\right),$$

where

$$c = \frac{2\left(\frac{3}{4}\|W''\|_\infty + 18\right)\|H(0)\|_2}{\|W'\|_\infty}\left(\frac{e^{\frac{T}{2}\|W'\|_\infty} - 1}{T}\right) + \frac{3}{2}\|W'\|_\infty.$$

Sketch of the Proof We follow the argument as in the proof of the previous theorem. In fact, keeping similar notation, one gets

$$\left\|\Phi_n'(t)\right\| \leq \delta_m(n) + \sum_{|e|\leq 1} \int_0^t f(s) \left\|\Phi_{n+e}'(s)\right\| ds,$$

where $f(s) = c_3 \left\|H(s)\right\|_2 + \frac{1}{2}c_2$, with $c_3 = \frac{1}{4}\left\|W''\right\|_\infty + 6$. Iteration implies that

$$\begin{aligned} \left\|\Phi_n'(t)\right\| &\leq \sum_{k=|n-m|}^{\infty} \frac{1}{k!}\left(3\int_0^t f(\tau)d\tau\right)^k \\ &\leq \sum_{k=|n-m|}^{\infty} \frac{g(t)^k}{k!} \leq \left[\frac{g(t)}{|n-m|}\right]^{|n-m|} e^{|n-m|}e^{g(t)}, \quad (2) \end{aligned}$$

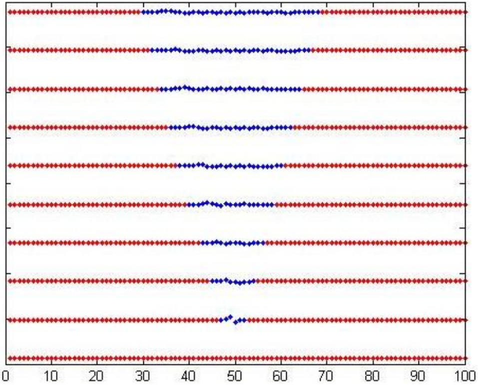
where $g(t) = \frac{3c_1c_3}{c_2}(e^{c_2|t|}-1) + \frac{3}{2}c_2|t|$. Given any $\mu > 0$. Then as we argued in the previous theorem one can show that

$$\left\|\Phi_n'(t)\right\| \leq e^{-\mu(|n-m|-c(e^{\mu+1}+\frac{1}{\mu})|t|)},$$

where $c = \frac{3c_1c_3(e^{\delta c_2}-1)}{\delta c_2} + \frac{3}{2}c_2$.

Some Numerics

In this example we consider a perturbed Toda system in the finite volume and assume periodic boundary conditions. We take $N = 100$ and $W(x) = \frac{1}{2}\cos(2\pi x)$. $a_n(0) = 1/2\ \forall n$ and $b_n(0) = 0$ $\forall n \neq 50$, $b_{50}(0) = 1$. The values of $a_n(t)$, $1 \leq n \leq 100$, over the time interval $[0, 10]$ are plotted below.



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