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Automorphic Equivalence Within Gapped Phases

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based on joint work with

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Outline:

1. Ideas
2. Set-Up
3. Assumptions
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Quantum Phase Transitions

One of the most important questions in quantum statistical mechanics is to understand the nature of phase transitions. Finding models which exhibit this phenomena is also crucial. Results concerning long range order and Anderson localization address this question directly.

There has been much recent interest in the phase associated to gapped ground states.

Basic Question: What does it mean for two gapped ground states of a quantum system to belong to the same phase?

Some Heuristics

Let ψ_0 and ψ_1 be two gapped ground states of a given quantum system. This means that there are (finite range) Hamiltonians $H(0)$ and $H(1)$ for which ψ_0 and ψ_1 are corresponding ground state eigenvectors, and moreover, each of these Hamiltonians has a spectral gap above the ground state energy.

In the physics literature, it has been said that ψ_0 and ψ_1 belong to the same phase if there is a family of (finite range) Hamiltonians $H(s)$, continuous for $0 \leq s \leq 1$, for which $H(s)$ has a non-vanishing spectral gap above its ground state for all $0 \leq s \leq 1$. Here, of course, $H(0)$ and $H(1)$ are as above.

Our Goal

Our goal was to prove a theorem which provides some rigor to these ideas. We present sufficient conditions under which the above heuristics guarantee that, in finite volume, the two ground states are unitarily equivalent. Moreover, this unitary equivalence can be obtained as a flow corresponding to quasi-local interactions. Our results further show that, again under certain conditions, this flow converges to an automorphism in the thermodynamic limit. Hence, unitary equivalence of states in finite volume leads to an automorphic equivalence of states in the infinite volume context.

Quantum Spin Systems

In this talk, we restrict our attention to quantum spin systems defined over $\mathbb{Z}^d$. Associate to each $x \in \mathbb{Z}^d$ a finite dimensional Hilbert space $\mathcal{H}_x$. For any finite $\Lambda \subset \mathbb{Z}^d$, a composite Hilbert space is given by

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x \,,$$

and the corresponding algebra of observables is

$$\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x) = \mathcal{B}(\mathcal{H}_\Lambda) \,.$$

Infinite Volume Algebras

If $X \subset \Lambda$, then by identifying $A \in \mathcal{A}_X$ with $A \otimes \mathbb{1}_{\Lambda \setminus X} \in \mathcal{A}_\Lambda$, we see that $\mathcal{A}_X \subset \mathcal{A}_\Lambda$. In this case, we can inductively define

$$\mathcal{A}_{\mathrm{loc}} = \bigcup_{\Lambda \subset \mathbb{Z}^d} \mathcal{A}_\Lambda \,,$$

the algebra of all local observables. The completion of $\mathcal{A}_{\mathrm{loc}}$ with respect to the operator norm, which we denote by $\mathcal{A}_{\mathbb{Z}^d}$, is the C^*-algebra of all quasi-local observables.

Models and Interactions

The models we are interested in are defined in terms of interactions. An interaction for these quantum spin systems is a map $\Phi : \mathcal{P}(\mathbb{Z}^d) \to \mathcal{A}_{\mathrm{loc}}$, i.e. from the set of finite subsets of $\mathbb{Z}^d$ to the algebra of local observables, such that for all finite $X \subset \mathbb{Z}^d$, $\Phi(X)^* = \Phi(X)$ and $\Phi(X) \in \mathcal{A}_X$.

For finite $\Lambda \subset \mathbb{Z}^d$, local Hamiltonians are given by

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X),$$

and the corresponding Heisenberg dynamics, $\{\tau_t^\Lambda\}_{t\in\mathbb{R}}$, is defined by

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}, \quad \text{for } A \in \mathcal{A}_\Lambda.$$

Assumptions

Let $\Phi(s)$ be a family of interactions on $\mathbb{Z}^d$ parametrized by $0 \leq s \leq 1$. Let $a > 0$.

1) We need a boundedness assumption on $\Phi(s)$:

$$\|\Phi\|_a = \sup_{x,y\in\mathbb{Z}^d} e^{a|x-y|} \sum_{\substack{X\subset\mathbb{Z}^d: \\ x,y\in X}} \sup_{0\leq s\leq 1} \|\Phi_X(s)\| < \infty$$

This clearly includes finite range, uniformly bounded interactions depending smoothly on s.

Assumption 1 implies a uniform Lieb-Robinson bound for $\Phi(s)$:

$$\left\| \left[\tau_t^{H_\Lambda(s)}(A), B \right] \right\| \leq C(A,B) e^{-a(d(X,Y)-v_a|t|)}$$

for all $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $t \in \mathbb{R}$.

Assumption (cont.)

2) We need a smoothness assumption on $\Phi(s)$:

$$\|\partial \Phi\|_a = \sup_{x,y\in\mathbb{Z}^d} e^{a|x-y|} \sum_{\substack{X\subset\mathbb{Z}^d:\\ x,y\in X}} |X| \sup_{0\le s\le 1} \|\Phi'_X(s)\| < \infty$$

3) We need a uniform gap assumption on $\Phi(s)$:

There exists $\gamma > 0$ s.t. to each finite $\Lambda \subset \mathbb{Z}^d$,

$$\sigma\left(H_\Lambda(s)\right) = \sigma_1\left(H_\Lambda(s)\right) \cup \sigma_2\left(H_\Lambda(s)\right)$$

each non-empty with

$$d\left(\sigma_1\left(H_\Lambda(s)\right), \sigma_2\left(H_\Lambda(s)\right)\right) \geq \gamma > 0.$$

Here γ, $C(A,B)$, and v_a are each independent of both Λ and s.

Gaps and the Spectral Projections

It is well-known that, in general, there exist unitaries $U_\Lambda(s)$ such that the spectral projection $P_\Lambda(s)$ corresponding to $\sigma_1(H_\Lambda(s))$ is given by

$$P_\Lambda(s) = U_\Lambda(s) P_\Lambda(0) U_\Lambda(s)^* .$$

In particular, this shows that the spectral projection $P_\Lambda(1)$ is unitarily equivalent to the spectral projection $P_\Lambda(0)$. Crucial for our result is that we have an explicit expression for $U_\Lambda(s)$. It is found, see talk of Sven, by writing the spectral projection as a contour integral over the resolvent.

The Spectral Flow

We show that $U_\Lambda(s)$ satisfies a specific evolution equation:

$$-i\frac{d}{ds}U_\Lambda(s) = D_\Lambda(s)U_\Lambda(s), \quad U_\Lambda(0) = \mathbb{1}$$

where

$$\begin{aligned} D_\Lambda(s) &= \int_{-\infty}^{\infty} \tau_t^{H_\Lambda(s)}\left(H'_\Lambda(s)\right) W_\gamma(t)\, dt \\ &= \sum_{X\subset\Lambda} \int_{-\infty}^{\infty} \tau_t^{H_\Lambda(s)}\left(\Phi'_X(s)\right) W_\gamma(t)\, dt\,. \end{aligned}$$

We define the spectral flow by setting

$$\alpha_s^\Lambda(A) = U_\Lambda(s)^* A U_\Lambda(s) \quad \text{for } A \in \mathcal{A}_\Lambda \quad \text{and } 0 \le s \le 1.$$

As is clear from the above, α_s^Λ is a flow corresponding to an s-dependent, quasi-local interaction.

A Lieb-Robinson Bound

A key result is the proof of a Lieb-Robinson type bound, uniform in the volume, for the spectral flow.

Theorem

Under assumptions 1-4 above,

$$\left\| \left[\alpha_s^\Lambda(A), B \right] \right\| \leq C(A,B) e^{v|s|} \sum_{x \in X, y \in Y} F(|x-y|)$$

for all $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, *and* $0 \leq s \leq 1$.

Here, for $r >> 1$, the function

$$F(r) = Ce^{-\mu \frac{r}{\ln^2(r)}} .$$

Up to some non-trivial technicalities, this bound follows from known results on LRBs for time dependent interactions.

Thermodynamic Limit

It is well-known that if the finite volume dynamics corresponding to an interaction over a quantum spin system satisfies a Lieb-Robinson bound which is independent of the volume, then there is a well defined dynamics in the thermodynamic limit. The same holds true for the spectral flow.

Theorem

Let $\{\Lambda_n\}$ be a sequence of finite, increasing, and exhaustive subsets of $\mathbb{Z}^d$ which satisfy additional technical assumptions. Then, there is a strongly continuous cocycle of $*$-automorphisms α_s defined on $\mathcal{A}_{\mathbb{Z}^d}$ such that for all $A \in \mathcal{A}_{\mathrm{loc}}$

$$\lim_{n\to\infty} \|\alpha_s^{\Lambda_n}(A) - \alpha_s(A)\| = 0\,.$$

The convergence is uniform for $s \in [0,1]$.

Properties of the Limiting Flow

The above result demonstrates that the unitary equivalence of the "isolated" spectral projections in the finite volume leads to an automorphism of the quasi-local observables in the thermodynamic limit. More is true:

Theorem

Under the assumptions above,

i) α_s *satisfies a sub-exponential Lieb-Robinson bound.*

ii) If β *is a local symmetry of the family* $\Phi(s)$*, i.e.* $\beta(\Phi_X(s)) = \Phi_X(s)$*, then* β *is also a symmetry of* α_s*, i.e.,* $\alpha_s \circ \beta = \alpha_s$.

iii) If the family $\Phi(s)$ *is translation invariant, then* α_s *commutes with translations.*

Equivalence of Gapped Quantum Phases

Under the assumptions above, let $\mathcal{S}_{\Lambda_n}(s)$ denote the set of states of the system in volume Λ_n that are mixtures of eigenstates with energies in $\sigma_1(H_{\Lambda_n}(s))$. Let $\mathcal{S}(s)$ be the set of weak-$*$ limit points as $n \to \infty$. For the finite volume, the definition of the spectral flow implies that

$$\mathcal{S}_{\Lambda_n}(s) = \mathcal{S}_{\Lambda_n}(0) \circ \alpha_s^{\Lambda_n} .$$

We have proven that

Theorem

The states $\omega(s) \in \mathcal{S}(s)$ are automorphically equivalent to $\omega(0) \in \mathcal{S}(0)$ for all $0 \leq s \leq 1$, in fact,

$$\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_s .$$

In Summary

Consider two gapped ground states of a given quantum spin system. If there is a connecting family of interactions $\Phi(s)$ which satisfy requisite boundedness, smoothness, and gap assumptions, then the ground state structure is preserved along this interpolating curve of models. In this case, it is reasonable to say that the two initial gapped ground states are in the same phase.

Note that the phase is being defined through an equivalence of states (or sets of states), not models. Moreover, the equivalence does not guarantee that the states in a set are automorphically equivalent among themselves.

Conclusion

We continue to improve our understanding of the dynamics corresponding to quantum many body systems. Recent results have been used to better understand correlations and low lying excitations. Moreover, locality properties have led to new perspectives on perturbation theory. However, there are a wealth of open questions yet to be explored . . .