Ferromagnetic Quantum Spin Chains with $SU_q(2)$ Symmetry

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Based on joint work with Bruno Nachtergaele and Stephen Ng

Spin-$1/2$, **spin chain**

1. Chain of length L

2. For each $k \in \{1, \ldots, L\}$,
   single site Hilbert space

   $$\mathcal{H}_k = \mathbb{C}^2, \quad \text{O.n. basis } |\uparrow\rangle, |\downarrow\rangle$$

3. Usual spin-1/2 spin matrices

   $$S^{(1)} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ S^{(2)} = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$
   $$S^{(3)} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

4. Total Hilbert space $\mathcal{H}_{[1,L]} = \bigotimes_{k=1}^{L} \mathcal{H}_k$,

   $$S_k^{(a)} = \mathbb{1}_{\mathcal{H}_1} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_{k-1}} \otimes S^{(a)} \otimes \mathbb{1}_{\mathcal{H}_{k+1}} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_L}$$

Heisenberg ferromagnet

$$H_{[1,L]} = \sum_{k=1}^{L-1} h_{k,k+1}$$

$$h_{k,k+1} = \frac{1}{4}\mathbb{1} - \boldsymbol{S}_k \cdot \boldsymbol{S}_{k+1}$$

$$= \frac{1}{4}\mathbb{1} - S_k^{(1)} S_{k+1}^{(1)} - S_k^{(2)} S_{k+1}^{(2)} - S_k^{(3)} S_{k+1}^{(3)}$$

In the $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$ basis for $\mathcal{H}_k \otimes \mathcal{H}_{k+1}$,

$$h_{k,k+1} = \frac{1}{2}\begin{bmatrix} 0 & & & \\ & +1 & -1 & \\ & -1 & +1 & \\ & & & 0 \end{bmatrix}$$

In other words,

$$h_{k,k+1} = \mathrm{Proj}\left(\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}\right)$$

Symmetries

Total spin operators: $a = 1, 2, 3$,

$$S^{(a)}_{[1,L]} = \sum_{k=1}^{L} S^{(a)}_k \, .$$

Call $S^{(3)}_{[1,L]}$ the "magnetization" operator.

The total spin operator

$$\boldsymbol{S}^2_{[1,L]} = (S^{(1)}_{[1,L]})^2 + (S^{(2)}_{[1,L]})^2 + (S^{(3)}_{[1,L]})^2$$

The triple,

Hamiltonian, magnetization, total spin

$$H_{[1,L]} \, , \quad S^{(3)}_{[1,L]} \, , \quad \boldsymbol{S}^2_{[1,L]}$$

commute.

A ferromagnetic Lieb-Mattis theorem:
For each $s \in \{\frac{1}{2}L, \frac{1}{2}L - 1, \ldots, \frac{1}{2} \text{ or } 0\}$, let

$$\mathcal{H}^{(s)}_{[1,L]} = \{\psi \in \mathcal{H}_{[1,L]} : \boldsymbol{S}^2_{[1,L]}\psi = s(s+1)\psi\}$$

Define

$$E_0(s) = \min\left\{\frac{\langle\psi\,,\ H_{[1,L]}\psi\rangle}{\|\psi\|^2} : 0 \neq \psi \in \mathcal{H}^{(s)}_{[1,L]}\right\}.$$

Then

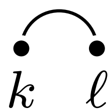
$$E_0\left(\frac{1}{2}L\right) < E_0\left(\frac{1}{2}L - 1\right) < \ldots < E_0\left(\frac{1}{2} \text{ or } 0\right)$$

$\star$ Lieb and Mattis proved the opposite ordering for the energy levels of bipartite antiferromagnets, in general.

Elements of the proof:

First proof: Nachtergaele, Spitzer, S
New proof: Nachtergaele, Ng, S

Given any $k < \ell$, define the spin singlet

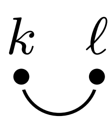


$$= \overset{\uparrow}{\underset{k}{\bullet}}\ \ \overset{\downarrow}{\underset{\ell}{\bullet}} \ - \ \overset{\downarrow}{\underset{k}{\bullet}}\ \ \overset{\uparrow}{\underset{\ell}{\bullet}}$$

$$= |{\uparrow}\rangle_{\mathcal{H}_k} \otimes |{\downarrow}\rangle_{\mathcal{H}_\ell} - |{\downarrow}\rangle_{\mathcal{H}_k} \otimes |{\uparrow}\rangle_{\mathcal{H}_\ell}\,.$$

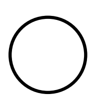
A spin configuration above a dot is understood as a vector.

Below = a dual vector, linear functional. Funny convention:



$$= \underset{\downarrow}{\overset{k}{\bullet}}\ \ \underset{\uparrow}{\overset{\ell}{\bullet}} \ - \ \underset{\uparrow}{\overset{k}{\bullet}}\ \ \underset{\downarrow}{\overset{\ell}{\bullet}}$$

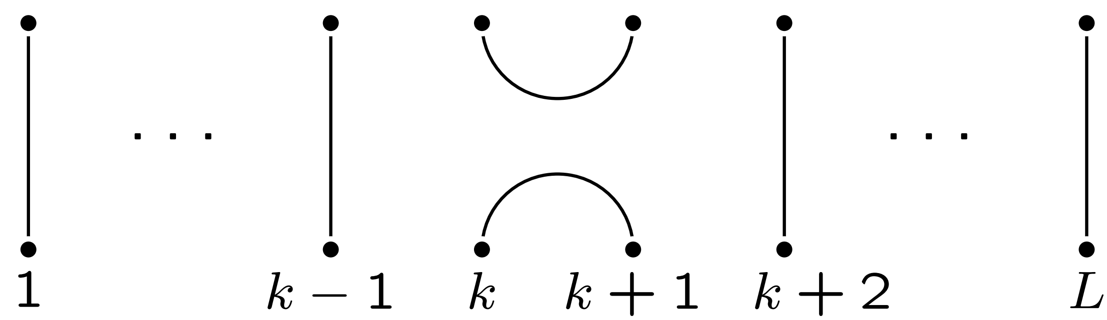
$$= \langle{\downarrow}|_{\mathcal{H}_k} \otimes \langle{\uparrow}|_{\mathcal{H}_\ell} - \langle{\uparrow}|_{\mathcal{H}_k} \otimes \langle{\downarrow}|_{\mathcal{H}_\ell}\,.$$



$$= -2$$

Temperley-Lieb algebra

With this, we may also define operators $U_{k,k+1} = -2h_{k,k+1}$



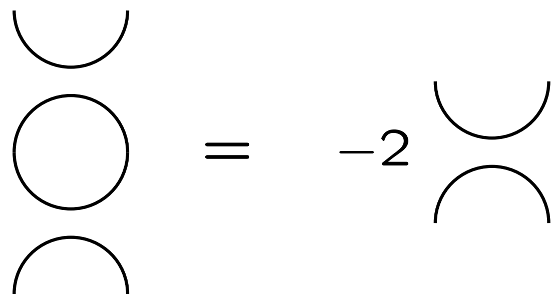
These satisfy the Temperley-Lieb algebra relations

$$U^2_{k,k+1} = -2U_{k,k+1}$$

$$U_{k,k+1}U_{k+1,k+2}U_{k,k+1} = U_{k,k+1}$$

$$U_{k,k+1}U_{k-1,k}U_{k,k+1} = U_{k,k+1}$$

$$|k-\ell| > 1 \;\Rightarrow\; U_{k,k+1}U_{\ell,\ell+1} = U_{\ell,\ell+1}U_{k,k+1}$$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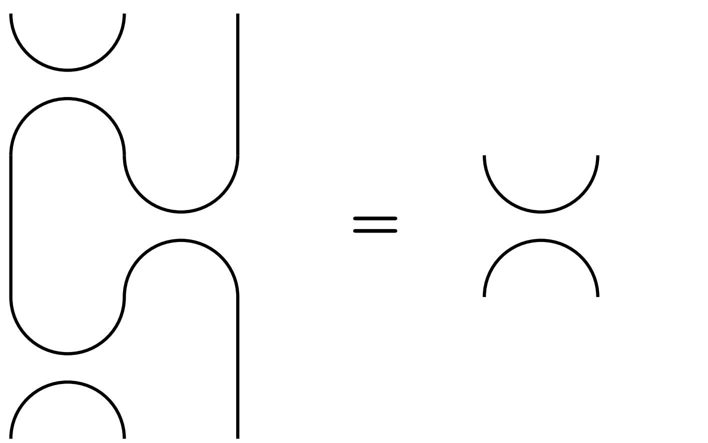
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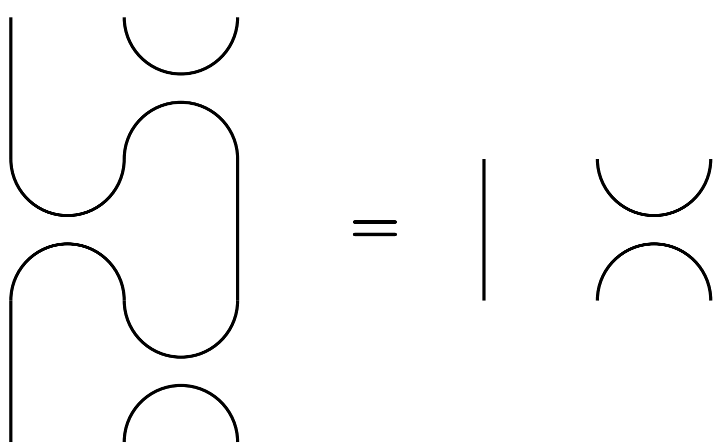
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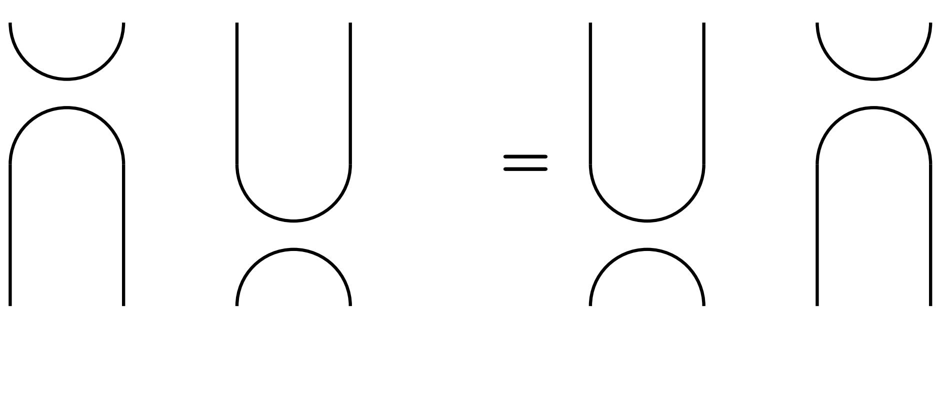
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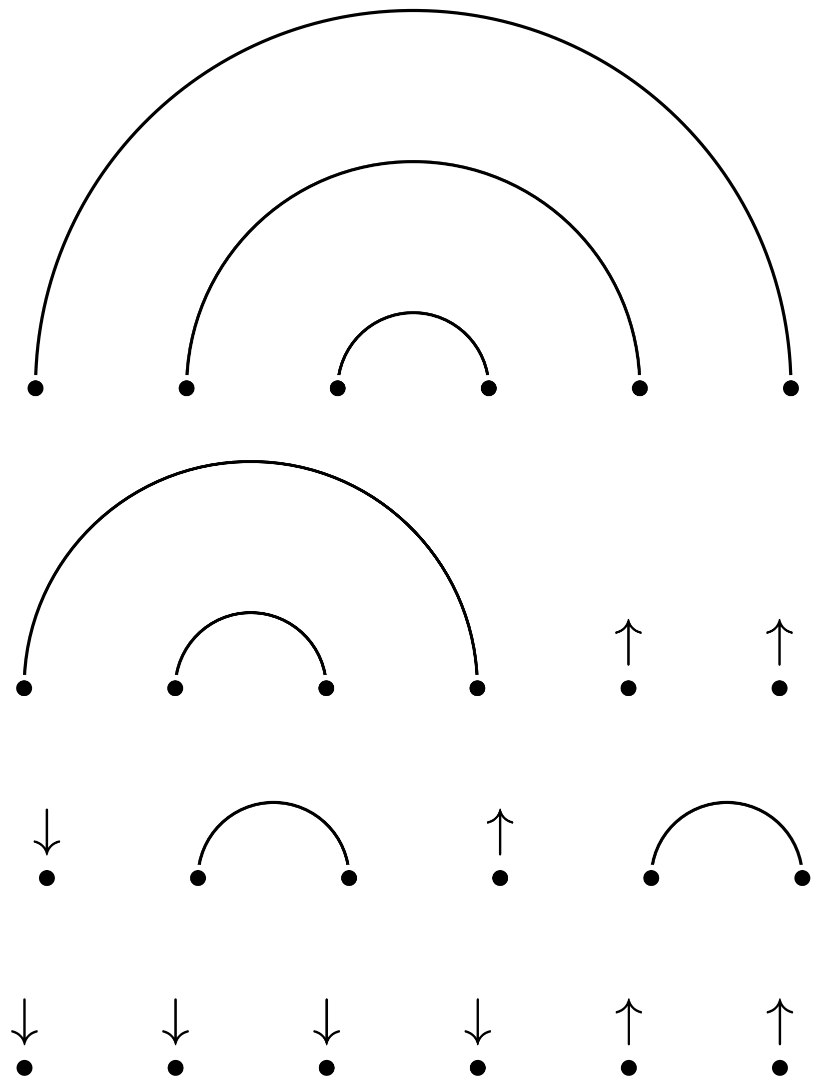
Graphical basis

- Each vertex has at most one arc incident to it.
- No two arcs cross.
- No arc spans a vertex with 0 arcs incident to it.
- All $\downarrow$ spins are to the left of all $\uparrow$ spins.

Examples:

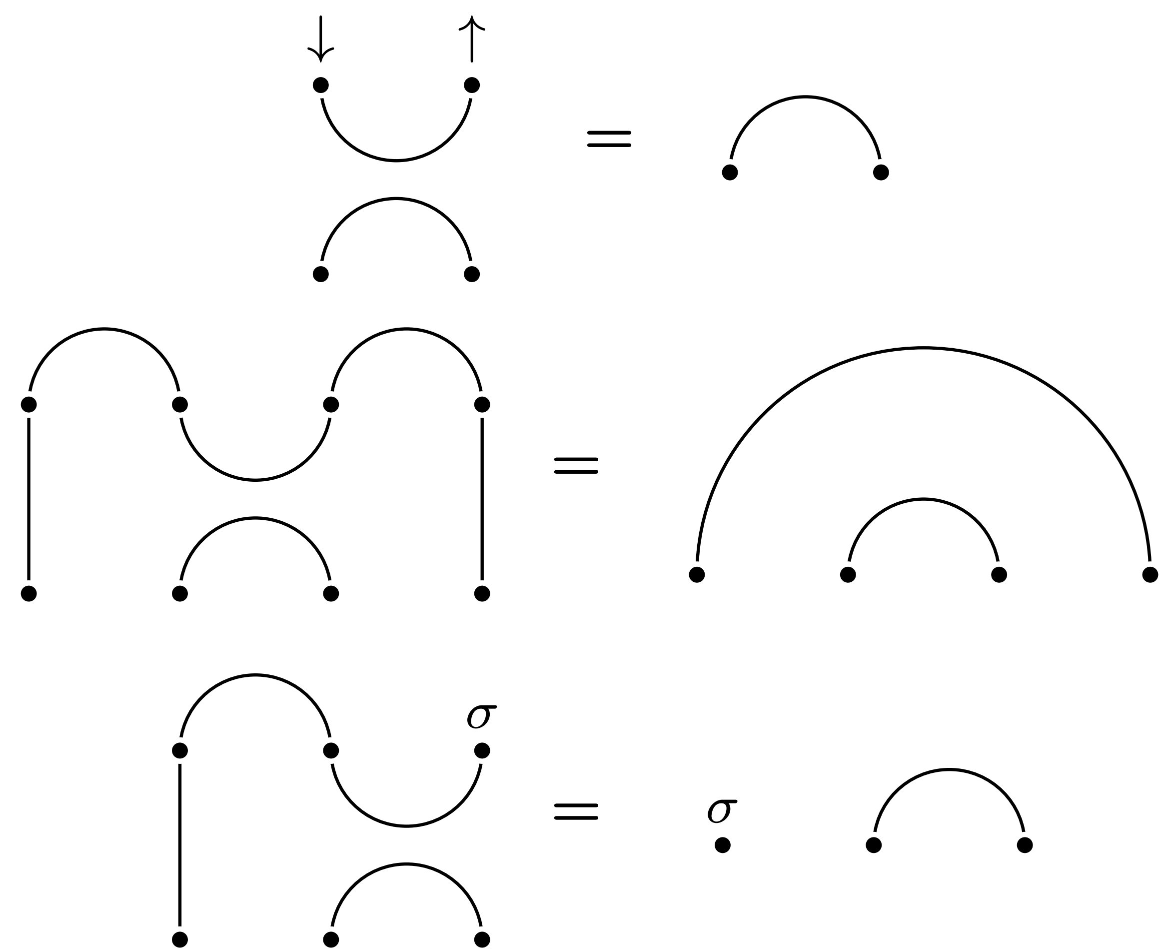


The set of all vectors satisfying these rules forms a non-orthonormal basis for $\mathcal{H}_{[1,L]}$.

Good signs

We want to apply the Perron-Frobenius theorem.

All $U_{k,k+1}$'s have nonnegative off-diagonal matrix entries in the graphical basis.



Highest weight vectors

The Hamiltonian has ergodic subspaces.

Suppose there are n arcs and all spins $= \uparrow$.

Then magnetization = total spin $= \frac{1}{2}L - n$.

These are called "highest weight vectors."

$\widetilde{\mathcal{H}}^{(s)}_{[1,L]}$:= span(h.w. vectors of total spin s)

$$E_0(s) = \min\left\{\frac{\langle\psi\,,\ H_{[1,L]}\psi\rangle}{\|\psi\|^2}\ :\ 0 \neq \psi \in \widetilde{\mathcal{H}}^{(s)}_{[1,L]}\right\}.$$

$$E_0(s) = -\max \operatorname{spec}(-H_{[1,L]} \restriction \widetilde{\mathcal{H}}^{(s)}_{[1,L]})$$

↑

P-F

Compare $E_0(s)$ and $E_0(s+1)$:
magnetization $= s$,
total spin $= s$ or $s+1$.

Includes subspace $\widetilde{\mathcal{H}}^{(s)}_{[1,L]}$

$n = \frac{1}{2}L - s$ arcs and all unpaired spins $\uparrow$

Also includes all graphical basis vectors

$n-1$ arcs and left-most unpaired spin $= \downarrow$

Changing the $\downarrow$ to $\uparrow$ defines a bijection

to h.w. vectors of $\widetilde{\mathcal{H}}^{(s+1)}_{[1,L]}$.

Matrix for $-H_{[1,L]}$:
nonnegative and block lower triangular

$$\begin{bmatrix} -H_{[1,L]} \upharpoonright \widetilde{\mathcal{H}}^{(s)}_{[1,L]} & 0 \\ * & -H_{[1,L]} \upharpoonright \widetilde{\mathcal{H}}^{(s+1)}_{[1,L]} \end{bmatrix}$$

Moreover there is a positive eigenvector: Apply spin lowering to the P-F eigenvector of

$$-H_{[1,L]} \upharpoonright \widetilde{\mathcal{H}}^{(s+1)}_{[1,L]}$$

Perron-Frobenius then implies

$$\Rightarrow \qquad \text{max spec}(-H_{[1,L]} \upharpoonright \widetilde{\mathcal{H}}^{(s+1)}_{[1,L]}) \\ \geq \text{max spec}(-H_{[1,L]} \upharpoonright \widetilde{\mathcal{H}}^{(s)}_{[1,L]})$$

$$E_0(s+1) \leq E_0(s)\,.$$

Two extensions

- Easy one.
  We need total ordering of spin sites.
  Can replace SU(2) by $\mathrm{SU}_q(2)$, $q > 0$.

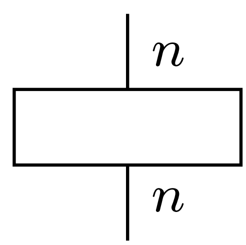
  $$\bigcirc = -[2] = -(q + q^{-1})$$

  XXZ model.

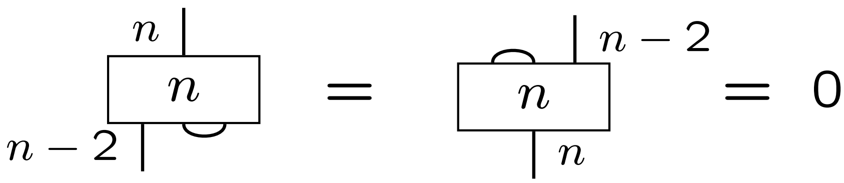
- Harder one.
  We can also extend to single-site spins $s > 1/2$.

Jones-Wenzl projection

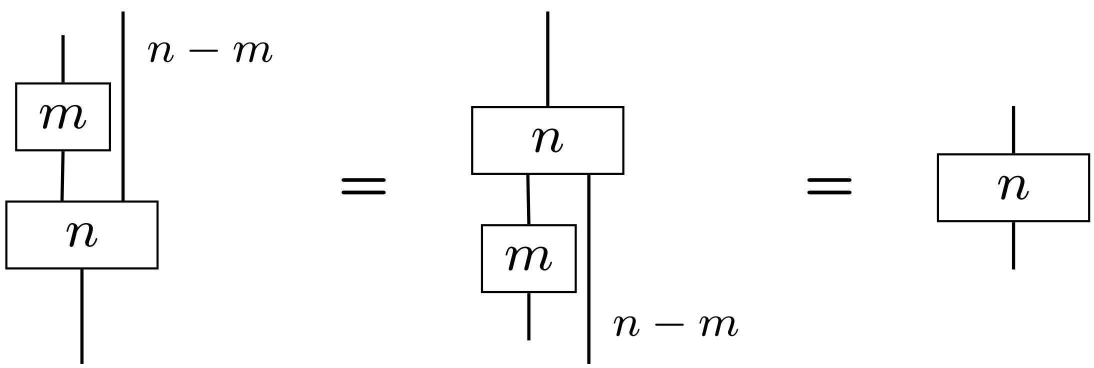
For n spins, the Jones-Wenzl projector is the projector onto symmetric tensors:



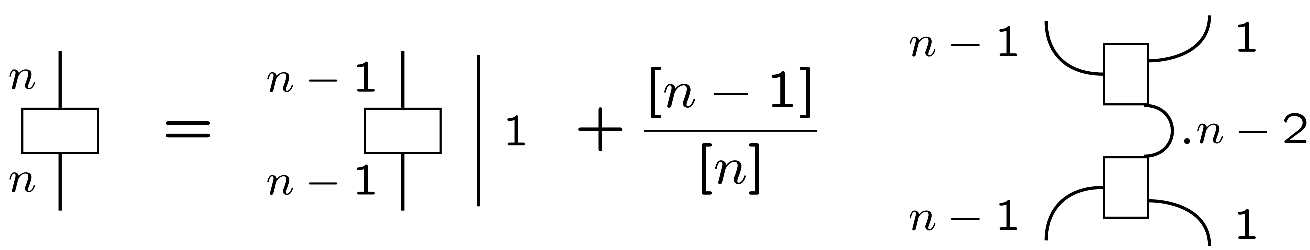
Identities



For $m \leq n$,



Jones-Wenzl relation:

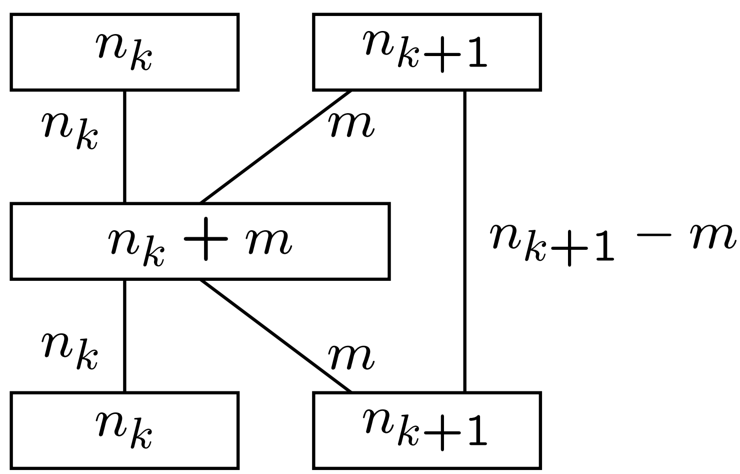


. . . and many more.

Positive interactions

DEFINE: interaction is positive,
if off-diagonal matrix entries ≥ 0
in the graphical "dual canonical" basis.
(See Frenkel and Khovanov.)

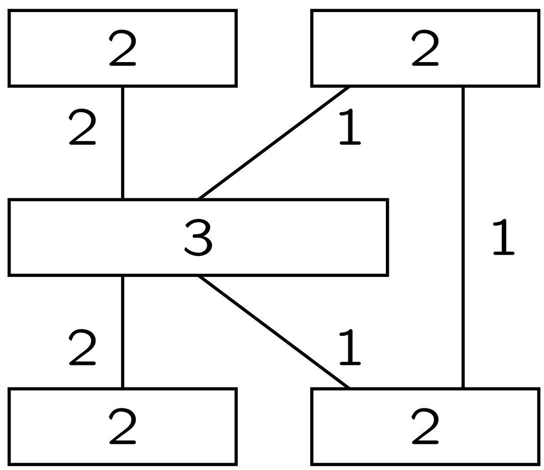
We characterized all positive interactions:
If $s_k = n_k/2$, $s_{k+1} = n_{k+1}/2$,
and $n_k \geq n_{k+1}$,



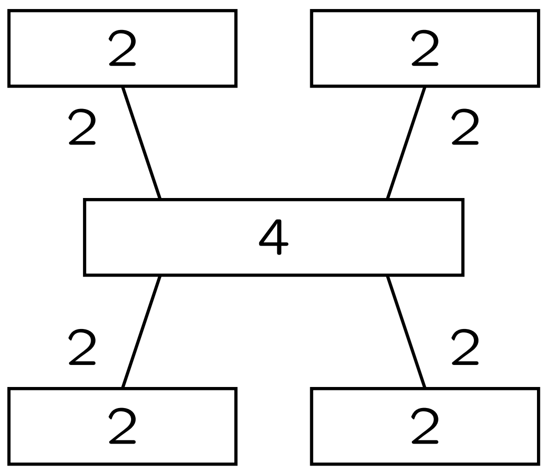
for $m = 0, \ldots, n_{k+1}$,
span the simplicial cone of positive interactions.

Examples

For $s = 1$, $n = 2s = 2$,



This is the Heisenberg antiferromagnet.

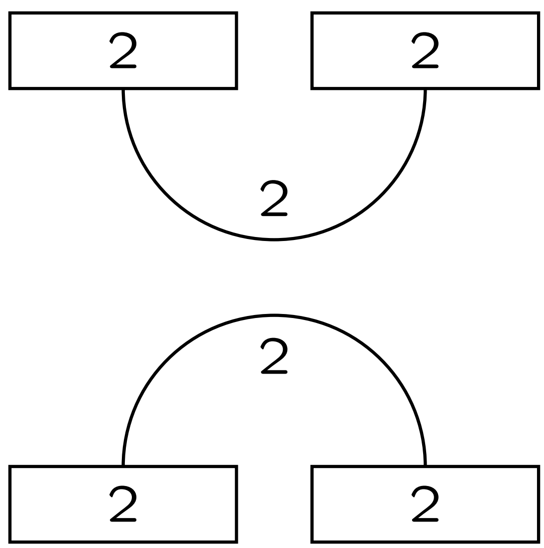


= the AKLT model, Affleck, Kennedy, Lieb, Tasaki.

Necessity?

The Perron-Frobenius theorem gives a sufficient condition for "ferromagnetic ordering of energy levels."

But it may not be necessary.



has $E_0(s) = 0$ for all s if L is big enough.