Lieb-Robinson bounds and Existence of the Thermodynamic Limit for a Class of Irreversible Quantum Dynamics

Anna Vershynina
co-authors: Bruno Nachtergaele, Valentin Zagrebnov

Harvard
May 19, 2011

Set up

Γ is a set of vertices $x \in \Gamma$ with a metric d.
The *Hilbert space* at $x \in \Gamma$ is $\mathcal{H}_x$ and for any finite $\Lambda \subset \Gamma$ is

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x.$$

The *algebra of observables* supported in Λ is

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x).$$

The algebra of local observables is

$$\mathcal{A}_\Gamma^{\mathrm{loc}} = \bigcup_{\Lambda\subset\Gamma} \mathcal{A}_\Lambda.$$

$\mathcal{A}_\Gamma$ is the norm completion of $\mathcal{A}_\Gamma^{\mathrm{loc}}$.

Restriction on Γ: there exists a non-increasing function such that:
i) F is uniformly integrable over Γ, i.e.,

$$\|F\| := \sup_{x\in\Gamma} \sum_{y\in\Gamma} F(d(x,y)) < \infty,$$

ii) F satisfies

$$C := \sup_{x,y\in\Gamma} \sum_{z\in\Gamma} \frac{F(d(x,z))F(d(y,z))}{F(d(x,y))} < \infty.$$

For any $\mu > 0$ the function

$$F_\mu(x) = e^{-\mu x} F(x),$$

also satisfies i) and ii) with $\|F_\mu\| \leq \|F\|$ and $C_\mu \leq C$.

- Hamiltonian part: for every $t \in \mathbb{R}$

$$\Phi(t, \cdot) : \{\textit{set of subsets of } \Gamma\} \to \mathcal{A}_\Gamma,$$

  such that $\Phi(t, X) \in \mathcal{A}_X$ and $\Phi(t, X)^* = \Phi(t, X)$.
- Dissipative part: for every $t \in \mathbb{R}$ and any finite $Z \subset \Gamma$

$$L_a(t, Z) \in \mathcal{A}_Z, \ a = 1, ...N(Z).$$

- For every $t \in \mathbb{R}$ and any finite $\Lambda \subset \Gamma$ define the family of bounded linear maps $\mathcal{L}_\Lambda : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$:

$$\Psi_Z(t)(A) = i[\Phi(t, Z), A] + \sum_{a=1}^{N(Z)} (L_a^*(t, Z) A L_a(t, Z) - \frac{1}{2}\{L_a(t, Z)^* L_a(t, Z), A\})$$

$$\mathcal{L}_\Lambda(t)(A) = \sum_{Z \subset \Lambda} \Psi_Z(t)(A),$$

  for all $A \in \mathcal{A}_\Lambda$.

Assume that the maps $\Psi_Z(t)$ are *completely bounded*, that is, for any $n \geq 1$ the maps $\Psi_Z(t) \otimes \mathrm{id}_{M_n}$ are bounded with uniformly bounded norm.
Define the *cb-norm* as

$$\|\Psi\|_{\mathrm{cb}} = \sup_{n \geq 1} \|\Psi \otimes \mathrm{id}_{M_n}\| < \infty$$

Assumption

Given (Γ, d) and F as above assume that

1. *For all finite $\Lambda \subset \Gamma$, $\mathcal{L}_\Lambda(t)$ is norm-continuous in t.*
2. *There exists $\mu > 0$ such that for every $t \in \mathbb{R}$*

$$\|\Psi\|_{t,\mu} := \sup_{s \in [0,t]} \sup_{x,y \in \Lambda} \sum_{Z \ni x,y} \frac{\|\Psi_Z(s)\|_{\mathrm{cb}}}{F_\mu(d(x,y))} < \infty.$$

Note that

$$\|\mathcal{L}_\Lambda(t)\| \le \sum_{Z\subset\Lambda} \|\Psi_Z(t)\| \le \sum_{x,y\in\Lambda} \sum_{Z\ni x,y} \|\Psi_Z(t)\|_{\mathrm{cb}} \le \|\Psi\|_{t,\mu} |\Lambda| \|F\| := M_t.$$

Also one gets $M_s \le M_t$ for $s < t$.
Fix $T > 0$, for all $A \in \mathcal{A}_\Lambda$ and $t \in [0, T]$, let $A(t)$ be a solution to the initial value problem

$$\frac{d}{dt} A(t) = \mathcal{L}_\Lambda(t) A(t), \quad A(0) = A. \tag{1}$$

Note that

$$\|\mathcal{L}_\Lambda(t)\| \leq \sum_{Z\subset\Lambda} \|\Psi_Z(t)\| \leq \sum_{x,y\in\Lambda} \sum_{Z\ni x,y} \|\Psi_Z(t)\|_{\mathrm{cb}} \leq \|\Psi\|_{t,\mu}|\Lambda|\|F\| := M_t.$$

Also one gets $M_s \leq M_t$ for $s < t$.
Fix $T > 0$, for all $A \in \mathcal{A}_\Lambda$ and $t \in [0, T]$, let $A(t)$ be a solution to the initial value problem

$$\frac{d}{dt}A(t) = \mathcal{L}_\Lambda(t)A(t), \quad A(0) = A. \tag{1}$$

For $0 \leq s \leq t \leq T$, define $\{\gamma^\Lambda_{t,s}\}_{0\leq s\leq t} \subset \mathcal{B}(\mathcal{A}_\Lambda, \mathcal{A}_\Lambda)$ by

$$\gamma^\Lambda_{t,s}(A) = A(t),$$

where $A(t)$ is the unique solution of (1) for $t \in [s, T]$ with initial condition $A(s) = A$.

Theorem

Let $T > 0$, and for $t \in [0, T]$, let $\mathcal{L}(t)$ be a norm-continuous family of bounded linear operator on a C^*-algebra $\mathcal{A}$. Suppose that
(i) $\mathcal{L}(t)(\mathbb{1}) = 0$;
(ii) for all $A \in \mathcal{A}$, $\mathcal{L}(t)(A^*) = \mathcal{L}(t)(A)^*$;
(iii) for all $A \in \mathcal{A}$, $\mathcal{L}(t)(A^*A) - \mathcal{L}(t)(A^*)A - A^*\mathcal{L}(t)(A) \geq 0$;
then the maps $\gamma_{t,s}$, $0 \leq s \leq t \leq T$ are a norm-continuous cocycle of unit preserving completely positive maps.

The map $\gamma : \mathcal{A} \to \mathcal{B}$ is *completely positive* if, for all $n \geq 1$, the maps $\gamma \otimes \mathrm{id} : \mathcal{A} \otimes M_n \to \mathcal{B} \otimes M_n$ are positive.

Theorem

Let $T > 0$, and for $t \in [0, T]$, let $\mathcal{L}(t)$ be a norm-continuous family of bounded linear operator on a C^*-algebra $\mathcal{A}$. Suppose that
(i) $\mathcal{L}(t)(\mathbb{1}) = 0$;
(ii) for all $A \in \mathcal{A}$, $\mathcal{L}(t)(A^*) = \mathcal{L}(t)(A)^*$;
(iii) for all $A \in \mathcal{A}$, $\mathcal{L}(t)(A^*A) - \mathcal{L}(t)(A^*)A - A^*\mathcal{L}(t)(A) \geq 0$;
then the maps $\gamma_{t,s}$, $0 \leq s \leq t \leq T$ are a norm-continuous cocycle of unit preserving completely positive maps.

The map $\gamma : \mathcal{A} \to \mathcal{B}$ is *completely positive* if, for all $n \geq 1$, the maps $\gamma \otimes \mathrm{id} : \mathcal{A} \otimes M_n \to \mathcal{B} \otimes M_n$ are positive.
For $\mathcal{L}_\Lambda(t)$ property (iii), which is called *complete dissipativity*, follows from

$$\mathcal{L}(t)(A^*A) - \mathcal{L}(t)(A^*)A - A^*\mathcal{L}(t)(A) = \sum_{Z \subset \Lambda} \sum_{a=1}^{N(Z)} [A, L_a(t, Z)]^* [A, L_a(t, Z)] \geq 0.$$

Outline of the proof

Let $\mathcal{L}(t)$, $t \geq 0$ be a family of operators on a C^*-algebra $\mathcal{A}$ satisfying the assumptions of Theorem.
An Euler type approximation

$$T_n(t) = \prod_{k=n}^{1} \left(\mathrm{id} + \frac{t}{n} \mathcal{L}(\frac{kt}{n}) \right) .$$

Lemma

Uniformly for all $t \in [0, T]$,

$$\lim_{n \to \infty} \| T_n(t) - \gamma_{t,0} \| = 0 .$$

Then we show that $T_n(A^* A) \geq N_n(t) \|A\|$ and $N_n(t) \to 0$ as $n \to \infty$.

Lieb-Robinson bounds

For reversible dynamics:
there are constants $v, \mu > 0$ such that for $A \in A_X$ and $B \in \mathcal{A}_Y$,

$$\|[A, \tau_t(B)]\| \leq C(A, B)e^{-\mu(d(X,Y) - v|t|)}.$$

Lieb-Robinson bounds

For reversible dynamics:
there are constants $v, \mu > 0$ such that for $A \in A_X$ and $B \in \mathcal{A}_Y$,

$$\|[A, \tau_t(B)]\| \leq C(A, B)e^{-\mu(d(X,Y)-v|t|)}.$$

For $X \subset \Lambda$, let $\mathcal{B}_X$ be a subspace of $\mathcal{B}(\mathcal{A}_X)$ consisting of all completely bounded linear maps that vanish on $\mathbb{1}$.
Note that the operator

$$\mathcal{K}(B) = [A, B] + \sum_{a=1}^{N}(L_a^* B L_a - \frac{1}{2}\{L_a^* L_a, B\}),$$

where $A, L_a \in \mathcal{A}_X$, belongs to $\mathcal{B}_X$.

Theorem

Suppose Assumption holds. Then for $X, Y \subset \Lambda$ and any operators $\mathcal{K} \in \mathcal{B}_X$ and $B \in \mathcal{A}_Y$ we have that

$$\|\mathcal{K}(\gamma_{t,s}^{\Lambda}(B))\| \leq \frac{\|\mathcal{K}\|_{\mathrm{cb}} \, \|B\|}{C_\mu} e^{\|\Psi\|_{t,\mu} C_\mu |t-s|} \sum_{x \in X} \sum_{y \in Y} F_\mu(d(x,y)) \, .$$

It could be rewritten as follows

$$\|\mathcal{K}\gamma_{t,s}^{\Lambda}(B)\| \leq \frac{\|\mathcal{K}\|_{\mathrm{cb}}\|B\|}{C_\mu} \|F\| \min(|X|,|Y|) e^{-\mu\left(d(X,Y) - \frac{\|\Psi\|_{t,\mu} C_\mu}{\mu}(t-s)\right)}.$$

The Lieb-Robinson velocity of the propagation is

$$v_{t,\mu} := \frac{\|\Psi\|_{t,\mu} C_\mu}{\mu}.$$

Outline of the proof of Lieb-Robinson bound

Consider the function $f : [s, \infty) \to \mathcal{A}$,

$$f(t) = \mathcal{K}\gamma_{t,s}^{\Lambda}(B).$$

For $X \subset \Lambda$, let $X^c = \Lambda \setminus X$ and define

$$\begin{aligned} \mathcal{L}_{X^c}(t) &= \sum_{Z, Z\cap X=\emptyset} \mathcal{L}_Z(t) \\ \bar{\mathcal{L}}_X(t) &= \mathcal{L}_\Lambda(t) - \mathcal{L}_{X^c}(t). \end{aligned}$$

Then

$$\begin{aligned} f'(t) &= \mathcal{K}\mathcal{L}_\Lambda(t)\gamma_{t,s}^{\Lambda}(B) \\ &= \mathcal{L}_{X^c}(t)\mathcal{K}\gamma_{t,s}^{\Lambda}(B) + \mathcal{K}\bar{\mathcal{L}}_X(t)\gamma_{t,s}^{\Lambda}(B) \\ &= \mathcal{L}_{X^c}(t)f(t) + \mathcal{K}\bar{\mathcal{L}}_X(t)\gamma_{t,s}^{\Lambda}(B) \ , \end{aligned}$$

$\gamma_{t.s}^{X^c}$ is generated by $\mathcal{L}_{X^c}(t)$.
Then,

$$f(t) = \gamma_{t,s}^{X^c} f(s) + \int_s^t \gamma_{t,r}^{X^c} \mathcal{K} \bar{\mathcal{L}}_X(r) \gamma_{r,s}^{\Lambda}(B) dr$$

and therefore

$$\|f(t)\| \leq \|f(s)\| + \|\mathcal{K}\|_{\mathrm{cb}} \int_s^t \|\bar{\mathcal{L}}_X(r) \gamma_{r,s}^{\Lambda}(B)\| dr.$$

$\gamma_{t.s}^{X^c}$ is generated by $\mathcal{L}_{X^c}(t)$.
Then,

$$f(t) = \gamma_{t,s}^{X^c} f(s) + \int_s^t \gamma_{t,r}^{X^c} \mathcal{K} \bar{\mathcal{L}}_X(r) \gamma_{r,s}^{\Lambda}(B) dr$$

and therefore

$$\|f(t)\| \leq \|f(s)\| + \|\mathcal{K}\|_{\text{cb}} \int_s^t \|\bar{\mathcal{L}}_X(r) \gamma_{r,s}^{\Lambda}(B)\| dr.$$

Define

$$C_B(X,t) := \sup_{\mathcal{T} \in \mathcal{B}_X} \frac{\|\mathcal{T} \gamma_{t,s}^{\Lambda}(B)\|}{\|\mathcal{T}\|_{\text{cb}}}.$$

Clearly,

$$C_B(X,s) \leq \|B\| \delta_Y(X),$$

where $\delta_Y(X) = 0$ if $X \cap Y = \emptyset$ and $\delta_Y(X) = 1$ otherwise.

$$C_B(X,t) \le C_B(X,s) + \sum_{Z\cap X\neq\emptyset} \int_s^t \|\mathcal{L}_Z(s)\| C_B(Z,s)ds.$$

Iterating this inequality we find:

$$C_B(X,t) \le \|B\| \sum_{n=0}^{\infty} \frac{(t-s)^n}{n!} a_n ,$$

where:

$$a_n \le \|\Psi\|_{t,\mu}^n C_\mu^{n-1} \sum_{x\in X} \sum_{y\in Y} F_\mu(d(x,y)),$$

for $n \ge 1$ and $a_0 = 1$.
Therefore

$$C_B(X,t) \le \frac{\|B\|}{C_\mu} e^{\|\Psi\|_{t,\mu} C_\mu (t-s)} \sum_{x\in X} \sum_{y\in Y} F_\mu(d(x,y)).$$

From definition of $C_B(X,t)$ we get

$$\|\mathcal{K}(\gamma^{\Lambda}_{t,s}(B))\| \leq \frac{\|\mathcal{K}\|_{\mathrm{cb}}\,\|B\|}{C_\mu} e^{\|\Psi\|_{t,\mu} C_\mu |t-s|} \sum_{x\in X}\sum_{y\in Y} F_\mu(d(x,y))\,.$$

Using definition of F_μ we can rewrite it

$$\|\mathcal{K}\gamma^{\Lambda}_{t,s}(B)\| \leq \frac{\|\mathcal{K}\|_{\mathrm{cb}}\|B\|}{C_\mu}\|F\|\min(|X|,|Y|)e^{-\mu(d(X,Y)-\frac{\|\Psi\|_{t,\mu}C_\mu}{\mu}(t-s))}.$$

Existence of the Thermodynamic Limit

Now let Γ be an infinite set.
Let $\Lambda_n \subset \Gamma$, $n \geq 1$ be a family of an increasing and absorbing sequence of finite subsets. Suppose that Assumption (2) holds *uniformly* for all Λ_n.

Theorem

Suppose that Assumption holds for $\Lambda = \Gamma$. Then, there exists a strongly continuous cocycle of unit-preserving completely positive maps $\gamma^{\Gamma}_{t,s}$ on $\mathcal{A}_\Gamma$ such that for all $0 \leq s \leq t$,

$$\lim_{n\to\infty} \|\gamma^{\Lambda_n}_{t,s}(A) - \gamma^{\Gamma}_{t,s}(A)\| = 0, \tag{2}$$

for all $A \in \mathcal{A}_\Gamma$.

Outline of the proof

Consider the function

$$f(t) := \gamma_{t,s}^{(n)}(A) - \gamma_{t,s}^{(m)}(A)\ .$$

Calculate the derivative

$$\begin{aligned} f'(t) &= \mathcal{L}_n \gamma_{t,s}^{(n)}(A) - \mathcal{L}_m \gamma_{t,s}^{(m)}(A) \\ &= \mathcal{L}_n(t)(\gamma_{t,s}^{(n)}(A) - \gamma_{t,s}^{(m)}(A)) + (\mathcal{L}_n(t) - \mathcal{L}_m(t))\gamma_{t,s}^{(m)}(A) \\ &= \mathcal{L}_n(t) f(t) + (\mathcal{L}_n(t) - \mathcal{L}_m(t))\gamma_{t,s}^{(m)}(A). \end{aligned}$$

The solution to this differential equation is

$$f(t) = \int_s^t \gamma_{t,r}^{(n)}(\mathcal{L}_n(r) - \mathcal{L}_m(r))\gamma_{r,s}^{(m)}(A) dr.$$

Therefore

$$\|f(t)\| \le \int_s^t \|(\mathcal{L}_n(r) - \mathcal{L}_m(r))\gamma_{r,s}^{(m)}(A)\|dr$$
$$\le \sum_{z\in\Lambda_n\setminus\Lambda_m} \sum_{Z\ni z} \int_s^t \|\Psi_Z(r)(\gamma_{r,s}^{(m)}(A))\|dr.$$

Using the Lieb-Robinson bound and the exponential decay condition

$$\|f(t)\| \le \|A\|\|\Psi\|_{t,\mu} \int_s^t e^{\mu v_{r,\mu} r} dr |X| \sup_{x\in X} \sum_{z\in\Lambda_n\setminus\Lambda_m} F_\mu(d(x,z)).$$

Thus

$$\|(\gamma_{t,s}^{(n)} - \gamma_{t,s}^{(m)})(A)\| \to 0, \ as\ n, m \to \infty.$$

Thank you!