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Voronoi formulas and applications

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Poisson summation formula:

$$\sum_n f(n) = \sum_n \hat{f}(n)$$

for any Schwartz function f, here

$$\hat{f}(t) := \int_{\mathbb{R}} f(x) e^{-2\pi i x t} dx$$

is its Fourier transform.

Voronoi formulas are generalizations of Poisson summation formulas.

Motivations

1) Dirichlet's divisor problem:

Let $d(n) :=$ # of positive divisors of an integer n, Dirichlet proved

$$\sum_{n \le N} d(n) = N \log N + (2\gamma - 1) N + O(N^{\frac{1}{2}})$$

where

$$\gamma = 1 - \int_1^{\infty} \{y\} y^{-2} \, dy = 0.5772\ldots$$

is the Euler constant.

<u>Method of proof:</u> Switching divisors

hyperbola method

Motivations

2) Gauss circle problem

Let

$$r_2(n) := \#\{(x,y)\in\mathbb{Z}^2 \mid x^2+y^2=n\},$$

Gauss proved

$$\sum_{n\le x} r_2(n) = \pi x + O(\sqrt{x}).$$

<u>Method of proof:</u>

pack the circle with unit squares

let $A(x)$ be the area of the squares intersecting the boundary of the circle. then

$$\left|\sum_{n\le x} r_2(n) - \pi x\right| \le A(x) = O(\sqrt{x}).$$

Improve the error terms

Voronoi's formula for the divisor function:

For $g \in C_c^\infty(\mathbb{R}^+)$,

$$\sum_n d(n)\, g(n) = \int_0^\infty (\log x + 2\gamma)\, g(x)\, dx$$
$$- 2\pi \sum_n d(n) \int_0^\infty Y_0(4\pi\sqrt{nx})\, g(x)\, dx$$
$$+ 4 \sum_n d(n) \int_0^\infty K_0(4\pi\sqrt{nx})\, g(x)\, dx$$

where Y_0, K_0 are Bessel functions.

Using his formula, in 1903, 1904, Voronoi proved that the error term in the divisor problem can be improved to $O(N^{\frac{1}{3}+\varepsilon})$ while the conjecture of the sharp bound is $O(N^{\frac{1}{4}+\varepsilon})$.

5/ Voronoi also conjectured a similar formula for $r_2(n)$ which was proved by Hardy and Sierpinski:

$$\sum_n r_2(n)\, g(n) = \pi \int_0^\infty g(x)\, dx + \sum_n r_2(n)\, h(n)$$

here

$$h(y) = \pi \int_0^\infty g(x)\, J_0\left(2\pi \sqrt{xy}\right) dx$$

for any $g(x) \in C_c^\infty(\mathbb{R}^+)$.

This formula was used to improve the error term in the circle problem to $O(x^{\frac{1}{3}})$ while the best error term was conjectured to be $O(x^{\frac{1}{4}+\varepsilon})$.

Modularity

$$\sum_{n\geq 1} d(n) n^{-s} = \zeta^2(s)$$

$$\sum_{n\geq 1} r_2(n) n^{-s} = 4\zeta_{\mathbb{Q}(\sqrt{-1})}(s)$$

here $\zeta_{\mathbb{Q}(\sqrt{-1})}(s)$ is the Dedekind zeta function of the number fld $\mathbb{Q}(\sqrt{-1})$,

$$\zeta_{\mathbb{Q}(\sqrt{-1})}(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s}$$

which counts the # of ideals of norm n.

It factors

$$\zeta_{\mathbb{Q}(\sqrt{-1})}(s) = \zeta(s) L(s, \chi_4)$$

here χ_4 is the only primive character mod 4.

Modularity (Cont.)

Let

$$d_s(n) := \sum_{0<d|n} d^s,$$

$$E(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \mathrm{Im}(\gamma z)^s$$

its Fourier expansion

$$= \text{Constant term} + \frac{2\pi^s \sqrt{y}}{\Gamma(s)\zeta(2s)} \sum_{n \neq 0} d_{1-2s}(n)\, |n|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y) \cdot e^{2\pi i n x}.$$

$$\theta^2(z) = \sum_{0}^{\infty} r_2(n)\, e(nz)$$

with $e^{2\pi i z}$ denoted as $e(z)$.

$$\theta(z) = \sum_{-\infty}^{\infty} e(n^2 z)$$

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Automorphic forms for $\Gamma := SL(n,\mathbb{Z})$

Let $n \geq 2$,

$$\mathfrak{h}^n := GL(n,\mathbb{R}) / \langle O(n,\mathbb{R}) \cdot \mathbb{R}^\times \rangle$$

$$\cong \left\{ \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ & 1 & \cdots & x_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix} \;\middle|\; y_i > 0 \right\}$$

be the Iwasawa decomposition.

Def. An automorphic form f for Γ is a smooth function on $\mathfrak{h}^n$ satisfying

1) $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$, $z \in \mathfrak{h}^n$.

2) $Df = \lambda_D f$ for all invariant differential operators.

If $f \in L^2(\Gamma \backslash \mathfrak{h}^n)$ also satisfies

3) $\int_{\Gamma \cap U \backslash U} f(uz)\, d^*u = 0$ for all $z \in \mathfrak{h}^n$,

then f is called a Maass form for Γ.

Hecke operators

For every integer $m \geq 1$, we define a Hecke operator T_m acting on $L^2(\Gamma\backslash\mathfrak{h}^n)$:

$$T_m f(z) := \frac{1}{m^{\frac{n-1}{2}}} \sum_{\substack{\prod_{\ell=1}^{n} c_\ell = m \\ 0 \le c_{i,\ell} < c_\ell \\ (1 \le i < \ell \le n)}} f\left(\begin{pmatrix} c_1 & c_{12} & \cdots & c_{1n} \\ & c_2 & \cdots & c_{2n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} z\right).$$

These operators are normal operators and they commute with all the invariant differential operators.

<u>Def.</u> If f is a Maass cusp form for Γ as well as an eigenfunction of all Hecke operators T_m, we call it a Hecke Maass form.

Voronoi formula for GL(n)

<u>Thm</u> (Miller–Schmid). Let f be a Maass form for $SL(n,\mathbb{Z})$ with Fourier coeff. $a_{c_{n-2},\cdots,c_1,r}$, $(h,q)=1$, ϕ be a Schwartz function,

$$\sum_{r\neq 0} a_{c_{n-2},\cdots,c_1,r}\, e\left(\frac{-rh}{q}\right)\phi(r)$$

$$= |q| \sum_{d_1 \mid qc_1} \cdots \sum_{d_{n-2} \mid \frac{qc_1\cdots c_{n-2}}{d_1\cdots d_{n-3}}} \sum_{r\neq 0} \frac{a_{r,d_{n-2},\cdots,d_1}}{|rd_1\cdots d_{n-2}|}$$

$$\times S(r,\bar{h};q,c,d)\, \Phi\left(\frac{r d_{n-2}^2 d_{n-3}^2 \cdots d_1^{n-1}}{q^n c_{n-2} c_{n-3}^2 \cdots c_1^{n-2}}\right)$$

where

$$S(a,b;q,c,d) = \sum_{\substack{x_j \in \left(\mathbb{Z}/\frac{c_1\cdots c_j q}{d_1\cdots d_j}\mathbb{Z}\right)^* \\ j\le n-2}} e\left(\frac{d_1 x_1 a}{q} + \cdots + \frac{b\,\bar{x}_{n-2}}{\frac{c_1\cdots c_{n-2} q}{d_1\cdots d_{n-2}}}\right)$$

hyper Kloosterman sum.

Proof using the theory of automorphic distributions:

To illustrate Miller - Schmid's method, Let's consider the case $n=2$.

$$f(z) = \sum_{n\neq 0} a_n \sqrt{y}\, K_{it}(2\pi|ny|)\, e(nx).$$

the boundary value/distribution is

$$\tau(x) = \sum_{n\neq 0} c_n e^{2\pi i n x}$$

here $c_n = a_n n^{-it}$.

$\tau(x)$ satisfies

$$\tau(x) = |cx+d|^{1-2it}\, \tau\left(\frac{ax+b}{cx+d}\right).$$

Hence

$$\tau\left(x-\frac{d}{c}\right) = |cx|^{1-2it}\, \tau\left(\frac{a}{c}-\frac{1}{c^2x}\right)$$

Plug in the Fourier expansion & integrate against a function. ∎

Analytic applications

1. Sharp bounds for L^4 norm of Maass forms on $GL(2)$:

   Sarnak - Watson.

2. Cancellation in sums with additive twists:

Let a_n be the coeff. of a Maass form L-function on $GL(d)$:

$$S(N,\alpha) := \sum_{n \le N} a_n e(n\alpha)$$

here a_n is bounded on the average.

<u>Folklore conjecture:</u>

$$S(N,\alpha) \ll N^{\frac{1}{2}+\varepsilon}$$

uniformly in α.

<u>Folklore theorem</u> (known in 1960's by Chandrasekharan, Narasimhan, Selberg):

If $S(N,\alpha) = O_{\varepsilon,f}(N^{\beta+\varepsilon})$ for some $\frac{1}{2} \le \beta < 1$, then

$$\int_{-T}^{T} \left| L\left(\tfrac{1}{2}+it\right)\right|^2 dt = O_{\varepsilon,f}\left(T^{1+\varepsilon+(2\beta-1)d}\right).$$

Hence $\beta = 1/2$ gives the optimal bound $O_{\varepsilon,f}(T^{1+\varepsilon})$ for the second moment of the $GL(d)$ L-function.

i) $d = 2$ well known.

ii) $d = 3$ Miller showed $O_{\varepsilon,f}(N^{\frac{3}{4}+\varepsilon})$.
i.e. $\beta = 3/4$.
M. Young, Li and later Xiannan Li studied the dependence on f in the error term.

iii) Unknown.

Subconvexity bounds of L-functions

Suppose f is a Hecke-Maass form for $SL(n,\mathbb{Z})$, the standard L-function associated to f defined as follows

$$L(s,f) := \sum_{n\geq 1} \lambda_f(n) n^{-s}$$

is entire and satisfy the functional equation

$$\Lambda(s,f) = \varepsilon(f)\Lambda(1-s,\tilde{f})$$

where
$$\Lambda(s,f) := \pi^{-\frac{ns}{2}} \prod_{j=1}^{n} \Gamma\left(\frac{s+\lambda_j}{2}\right) L(s,f)$$

is the completed L-function,

$\varepsilon(f)$: root number with absolute value 1

$\tilde{f}$: the dual Maass form, i.e.,

$$\tilde{f}(z) = f(w_n {}^t z^{-1} w_n)$$

with
$$w_n = \begin{pmatrix} & & \pm 1 \\ & 1 & \\ \iddots & & \\ 1 & & \end{pmatrix}.$$

Analytic conductor:

$$q(s,f) := \prod_{j=1}^{n} \left(|s+\mu_j|+1\right)$$

Lindelöf hypothesis:

$$L(s,f) \ll_\varepsilon q(s,f)^{\varepsilon}$$

for $\mathrm{Re}\, s = \frac{1}{2}$.

Convexity bound

$$L(s,f) \ll q(s,f)^{\frac{1}{4}+\varepsilon}.$$

Weak subconvexity bound — Soundararajan

$$L(s,f) \ll \frac{q(s,f)^{\frac{1}{4}}}{[\log q(s,f)]^{A}}$$

here $A > 0$.

Subconvexity bound:

$$L(s,f) \ll q(s,f)^{\frac{1}{4}-\delta}$$

for some small $0 < \delta < \frac{1}{4}$.

Subconvexity bounds

1) $GL(2)$ L-functions: Well known.

2) Higher rank L-functions: Very little.

However they do have applications.

Example. Quantum unique ergodicity (QUE):
Let ϕ be a Hecke-Maass form for $SL(2,\mathbb{Z})$ normalized s.t. $\|\phi\|_{L^2}=1$. Then as $\lambda_\phi \to \infty$, the probability measure

$$d\mu_\phi(z) := |\phi(z)|^2 \frac{dx\,dy}{y^2}$$

weakly $*$ converges on $M = SL(2,\mathbb{Z})\backslash H$ to the normalized Poincare measure

$$d\mu(z) := \frac{3}{\pi}\frac{dx\,dy}{y^2},$$

i.e. for any smooth bounded function V on M,

$$\int_M V\,d\mu_\phi(z) \xrightarrow[\lambda_\phi \to \infty]{} \int_M V\,d\mu(z).$$

The Rudnick-Sarnak QUE conjecture was solved by Lindenstrauss using ergodic theory + trick of Soundararajan at the cusps. While the holomorphic analogue was solved by Holowinsky + Soundararajan using analytic method.

<u>Open</u>: <u>Effective rate of convergence.</u>

<u>Connection with L-functions:</u> <u>(Stronger QUE)</u>

By the spectral expansion:

$$L^2(M) = \underbrace{\mathbb{C}}_{\text{constant function}} \oplus \underbrace{C(M)}_{\text{Cuspidal spectrum}} \oplus \underbrace{\mathcal{E}(M)}_{\text{integrals of Eisenstein series}}$$

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We can assume the test function is a Maass cusp form since the other cases are similar and easier. The connection between QUE and L- functions is through

<u>Watson's formula</u>:

$$\left|\int_M V \, d\mu_\phi\right|^2 = C_M \cdot \frac{\Lambda(\frac{1}{2}, V\times\phi\times\phi)}{\Lambda(1, \mathrm{Sym}^2\phi)^2 \, \Lambda(1, \mathrm{Sym}^2 V)}$$

where $0 \neq C_M$ is a constant,

$\Lambda(\frac{1}{2}, V\times\phi\times\phi)$ is the value at $s=\frac{1}{2}$ of the completed Rankin triple product L- function. The denominator involves values at $s=1$ of the symmetric square L- function.

<u>Goal</u>: Show $\int_M V \, d\mu_\phi \to \int_M V \, d\mu = 0$ with an effective rate.

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$$\left|\int_M V \, d\mu_\phi\right|^2 \ll \frac{L(\frac{1}{2}, V\times\phi\times\phi)}{t_\phi \, L(1, \mathrm{sym}^2\phi)^2}$$

with $\lambda_\phi = \frac{1}{4} + t_\phi^2$.

It is known that

$$(\log t_\phi)^{-1} \ll L(1, \mathrm{sym}^2\phi) \ll \log t_\phi$$

and the convexity bound of $L(\frac{1}{2}, V\times\phi\times\phi)$ is t_ϕ which just fails to show

$$\int_M V \, d\mu_\phi \to 0.$$

A subconvexity bound for $L(\frac{1}{2}, V\times\phi\times\phi)$ with $t_\phi \to \infty$ and V being fixed would not only do the job but also gives us a polynomial decay of the period integral. Since

$$L(s, V\times\phi\times\phi) = L(s, \mathrm{sym}^2\phi\times V)\, L(s, V).$$

need:

Subconvexity bounds for $L(s, \mathrm{sym}^2\phi\times V)$.

Subconvexity bounds for Rankin-Selberg of $GL(3)\times GL(2)$

Stronger QUE requires a subconvexity bound for $GL(3)\times GL(2)$ L-functions with $GL(2)$ form fixed and $GL(3)$ form varying, this is still open. However in 2008, I proved subconvexity bounds for such $GL(3)\times GL(2)$ L-functions with $GL(2)$ form varying and $GL(3)$ form fixed.

<u>Thm</u> Let f be a fixed self dual Hecke-Maass form for $SL(3,\mathbb{Z})$ & u_j be an orthonormal basis of even Hecke-Maass for $SL(2,\mathbb{Z})$. then for $\varepsilon>0$, T large and $T^{\frac{3}{8}+\varepsilon}\le M\le T^{\frac{1}{2}}$. we have

$$\sum_j{}' e^{-\frac{(t_j-T)^2}{M^2}} L\left(\tfrac12, f\times u_j\right)+\frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-\frac{(t-T)^2}{M^2}}\left|L\left(\tfrac12-it, f\right)\right|^2 dt$$

$$\ll_{\varepsilon,f} T^{1+\varepsilon}M.$$

By a result of Lapid,

$$L(\tfrac{1}{2}, f \times u_j) \geqslant 0.$$

We have

<u>Cor 1.</u> $L(\tfrac{1}{2}, f \times u_j) \ll_{\varepsilon, f} (1+|t_j|)^{\frac{11}{8}+\varepsilon}$.

The convexity bound is $t_j^{\frac{3}{2}+\varepsilon}$, so the above is a subconvexity bound.

<u>Cor 2.</u> $L(\tfrac{1}{2}-it, f) \ll_{\varepsilon, f} (1+|t|)^{\frac{11}{16}+\varepsilon}$.

Convexity bound is $|t|^{\frac{3}{4}+\varepsilon}$, so the above is a subconvexity bound.

<u>Crucial tool:</u> GL(3) Voronoi formula first derived by Miller + Schmid.

<u>Other aspects:</u> Blomer, Munshi, ...

GL(4)

Suppose f is a Hecke-Maass form for $SL(4,\mathbb{Z})$ with Fourier coefficients $a_{m,n,k}$, the following problems are <u>open</u>:

1) Uniform cancellation in the additively twisted sum

$$\sum_{n\le N} a_{n,1,1}\, e(n\alpha)$$

for any $\alpha \in \mathbb{R}$.

2) Nontrivial bounds for

$$\sum_{n\le N} a_{n,1,1}\, e(2\sqrt{n}).$$

3) Subconvexity bounds for

$$L\left(\tfrac{1}{2}+it, f\right)$$

in the t-aspect and other aspects.

Another Version of the GL(4) Voronoi

<u>Thm</u> (Miller-L) Let $a_{r,d,n}$ be Fourier coefficients of a Maass form f for $SL(4,\mathbb{Z})$, $c \in \mathbb{Z}^+$, $(\lambda_1, \cdots, \lambda_4)$ be the Langlands parameters of f, $\delta_1 + \cdots + \delta_4 \equiv 0 \pmod 2$, $\phi(x) \in |x|^{-\lambda_1} \operatorname{sgn}(x)^{\delta_1} \mathcal{S}(\mathbb{R})$, define Φ as before. We have

$$\sum_{\substack{d \mid c \\ r \neq 0}} a_{r,d,1}\, d\, S\left(r, 1; \frac{c}{d}\right) \phi\left(\frac{d^2 r}{c^2}\right)$$

$$= \sum_{\substack{d \mid c \\ r \neq 0}} a_{1,d,r}\, d\, S\left(r, 1; \frac{c}{d}\right) \Phi\left(\frac{d^2 r}{c^2}\right)$$

with

$$S(k, l; c) = \sum_{\substack{x \pmod c \\ x\bar{x} \equiv 1\, (c)}} e\left(\frac{kx + l\bar{x}}{c}\right)$$

being the classical Kloosterman sum.

More applications of Voronoi formulas

Restriction theorems: studying L^p restriction norms to a submanifold proposed by Reznikov.

Many papers were written, however, most of them are on symmetric spaces of rank 1.

In 2011, M. Young and I proved a sharp upper bound for a $GL(3)$ Maass form restricted to a codimension 2 submanifold.

<u>Thm</u> (Young + L) Let F be a Hecke-Maass form for $SL(3,\mathbb{Z})$ in the tempered spectrum of Δ with eigenvalue λ_F and with normalized L^2 norm and first Fourier coefficient $A_F(1,1)$, we have

$$N(F) := \int_0^\infty \int_{SL(2,\mathbb{Z})\backslash \mathfrak{h}^2} \left| F\begin{pmatrix} z_2 y_1 & \\ & 1 \end{pmatrix} \right|^2 \frac{dx_2\, dy_2}{y_2^2} \frac{dy_1}{y_1}$$

$$\ll_\varepsilon \lambda_F^\varepsilon |A_F(1,1)|^2 .$$

<u>Remarks.</u> 1) Here

$$z_2 = \begin{pmatrix} 1 & x_2 \\ & 1 \end{pmatrix} \begin{pmatrix} y_2 & \\ & 1 \end{pmatrix} y_2^{-1/2}.$$

2) Temperedness means the archemedian factors of F satisfy the generalized Ramanujan conjecture.

3) In general, the size of $A_F(1,1)$ is not known: however, if F is self-dual, Ramakrishnan and Wang proved

$$A_F(1,1) \ll \log^{a} \lambda_F.$$

<u>Corollary.</u> If F is self dual,

$$N(F) \ll \lambda_F^{\varepsilon}.$$

<u>Crucial tool.</u> $GL(3)$ Voronoi formula first derived by Miller-Schmid.

Restriction theorems on $GL(n)$

<u>Thm</u> (S.C. Liu, M. Young + L) Assuming the generalized Lindelöf hypothesis for Rankin-Selberg L-functions, temperedness for all Hecke-Maass forms on $SL(m,\mathbb{Z})$ with $2 \le m \le n+1$ and the weighted local Weyl law, F is a Hecke Maass form for $SL(n+1,\mathbb{Z})$, we have

$$N(F) := \int_0^\infty \int_{SL_n(\mathbb{Z})\backslash \mathfrak{h}^n} \left| F\begin{pmatrix} z_2 y & \\ & 1 \end{pmatrix} \right|^2 d^* z_2 \frac{dy}{y}$$

$$\ll \lambda_F^{\varepsilon} |A_F(1)|^2 .$$

here

$$z_2 = \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ & 1 & \cdots & x_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_2 \cdots y_n & & & \\ & \ddots & & \\ & & y_2 & \\ & & & 1 \end{pmatrix} \prod_{k=2}^{n} y_k^{-\frac{n+1-k}{n}} ,$$

$A_F(1)$ is the first Fourier coefficient,

λ_F is the eigenvalue.

<u>Thm</u> (Liu, Young + L) Assuming temperedness of all Hecke-Maass forms on $SL(m,\mathbb{Z})$ with $2 \leq m \leq n+1$, and the weighted local Weyl law, and the spectral parameters of F satisfy the spacing condition $|\alpha_j - \alpha_k| \gg \lambda_F^{\varepsilon}$ for all $j \neq k$, we have

$$N(F) \gg_\varepsilon \lambda_F^{-\varepsilon}.$$

<u>Conjecture</u> (Liu, Young + L)

$$N(F) \sim C_n(\alpha) \log \lambda_F$$

where $C_n(\alpha)$ is a function of the spectral parameters of F which satisfies

$$C_n(\alpha) \asymp_n 1.$$