About Poincaré Duality

by Jacob Lurie

Let M be a manifold of dimension d, and let $q : E \to M$ be a Serre fibration of topological spaces, equipped with a section $s : M \to E$. For each open subset $U \subseteq M$, let $\mathrm{Sect}_c(U)$ denote the space of maps $M \to E$ which are sections of q and which agree with s outside of a compact subset of M.

Principle 1 (Nonabelian Poincare Duality)**.** *If the fibers of q are $(d-1)$-connected, then $\mathrm{Sect}_c(M)$ can be realized as the homotopy colimit of the diagram of spaces $\{\mathrm{Sect}_c(U)\}$, where U ranges over those open subsets of M which can be written as a finite disjoint union of disks.*

Example 2. Let A be an abelian group and let E be the product of M with an Eilenberg-MacLane space $K(A, n)$. Then we have canonical isomorphisms $\pi_i \mathrm{Sect}_c(M) \simeq \mathrm{H}_c^{n-i}(M; A)$. If M is an oriented manifold of dimension $d \leq n$, then the homotopy groups of the homotopy colimit of the diagram $\{\mathrm{Sect}_c(U)\}$ can be computed as the homology $\mathrm{H}_{i+d-n}(M; A)$, and Principle 1 recovers the statement of Poincare duality for the manifold M.

Example 3. Let G be a compact Lie group, let M be a compact manifold, and let E be the product of M with the classifying space of G. Then $\mathrm{Sect}_c(M)$ can be interpreted as a classifying space for G-bundles on the manifold M. Principle 1 implies that if G is simply connected and M has dimension ≤ 4 (or if G is connected and M has dimension ≤ 2), then we can reconstruct the homotopy type of this classifying space by studying G-bundles which have been trivialized outside a finite subset of M.

Problem 4 (Nonabelian Verdier Duality?)**.** *Formulate an analogue of Principle 1 which does not require the assumption that the map $q : E \to M$ be a Serre fibration.*