DEGENERATION OF NONABELIAN HODGE STRUCTURES

CARLOS SIMPSON

The Hodge structure on the cohomology of a complex Kähler manifold has turned out to be one of the most fertile and useful structures in complex geometry. Thanks to Wilfried Schmid's work, we have a very detailed and precise understanding of how the Hodge structures degenerate when the variety becomes singular, leading to a wide array of applications in many fields.

More recently, it has appeared useful to consider the "nonabelian cohomology" of a variety, whose first basic incarnation is the moduli space of flat bundles. A natural question is to try to generalize Wilfried's structure theorems on degenerations, to the nonabelian cohomology space. This was the subject of numerous discussions with Ludmil Katzarkov and Tony Pantev in the late 1990's. Results in this direction could have applications for the study of families of varieties in diverse contexts.

Suppose $(S, 0)$ is a pointed curve and $X \to S$ is a family of curves, whose general fibers are smooth and whose special fiber X_0 is reduced with simple normal crossings. Then we can consider the moduli space $M_{DR}(X/S) \to S$ of sheaves with integrable connections on the fibers. Over a general point $s \in S$, the fiber $M_{DR}(X_s)$ parametrizes flat bundles. It has a nonabelian Hodge structure where the analogue of the Hodge metric is Hitchin's hyperkähler metric. It degenerates to a moduli space $M_{DR}(X_0)$ of torsion-free sheaves on X_0 with logarithmic connections satisfying a compatibility condition at the crossings.

Problem: Understand the degeneration of the nonabelian Hodge structures on $M_{DR}(X_s)$ as $s \to 0$. We would like to have analogues of the nilpotent and SL_2 orbit theorems, and the norm estimates. These should give asymptotic information about the degeneration of the hyperkähler metric. There should be an analogue of the Clemens-Schmid exact sequence relating flat bundles on X_0 and the residue of the nonabelian Gauss-Manin or isomonodromic deformation connection. Look for a limiting nonabelian mixed Hodge structure.

One of the difficulties is to understand what happens near points in $M_{DR}(X_0)$ parametrizing torsion-free sheaves which are not locally free at the singularities.

References

1. E. Cattani, A. Kaplan, W. Schmid. Degeneration of Hodge structures. Annals of Math. 123 (1986), 457-535.
2. T. Chen. The associated map of the nonabelian Gauss-Manin connection. Central European J. of Math. 10 (2012), 1-15.
3. G. Daskalopoulos, R. Wentworth. The Yang-Mills flow near the boundary of Teichmüller space. Math. Ann. 318 (2000), 1-42.
4. N. Hitchin. The self-duality equations on a Riemann surface. Proc. London Math. Soc. 55 (1987), 59-126.
5. L. Katzarkov, T. Pantev, C. Simpson. Density of monodromy actions on non-abelian cohomology. Advances in Math. 179 (2003), 155-204.
6. W. Schmid. Variation of Hodge structure: the singularities of the period mapping. Inventiones 22 (1973), 211-319.

7. C. Simpson. The Hodge filtration on nonabelian cohomology. AMS Proc. Symp. Pure Math. 62 (1997), 217-284.
8. Y. Tsai. Nilpotent counterexamples to the non-abelian Clemens-Schmid exact sequence. Far East J. Math. Sci. (FJMS) 39 (2010), 193-208; see also "Non-abelian Clemens-Schmid Exact Sequences", Doctoral dissertation, University of California, Irvine (2003).

CNRS, University of Nice