What mathematics is really behind the distributional Γ-factors?

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This problem is motivated by my joint work with Wilfried Schmid on constructing L-functions via automorphic distributions (see [6–8]). An interesting – and poorly understood – aspect of our method is the structure of its archimedean integrals. In every case examined thus far, there is some change of coordinates that splits them into products of integrals of the form

$$G_\delta(s) \quad := \quad \int_{\mathbb{R}} e^{2\pi i x} \operatorname{sgn}(x)^\delta \, dx \quad = \quad i^\delta \, \frac{\Gamma_{\mathbb{R}}(s+\delta)}{\Gamma_{\mathbb{R}}(1-s+\delta)} \,, \quad \delta \in \{0,1\} \,, \tag{1}$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$ is the factor that famously accompanies $\zeta(s)$ in its functional equation. This identity is first proved by a contour shift when $0 < \operatorname{Re} s < 1$ (where it conditionally converges), and then extends to $s \in \mathbb{C}$ by meromorphic continuation.

Despite the uniformity of the answers we obtain, the computations have been performed by *ad hoc* combinatorial methods. I'd like to describe some examples here in the hopes that an appropriate algebraic context can be found to explain them. For that reason all integrals below will be expressed formally, without concern for convergence.

The first two examples are related to specific Γ-factor computations, while the last arose in understanding the existence and uniqueness of Whittaker functions on the group $GL(r,\mathbb{R})$. Both have vague resemblances to formulas from cluster algebras in [3]. What is really going on behind this, making it work?

Example 1: Rankin-Selberg tensor product on $GL(r) \times GL(r+1)$.

Let N and N_- be the subgroups of $r \times r$ unipotent upper and lower triangular matrices in $GL(r,\mathbb{R})$, respectively. Let $\psi(n) = e^{2\pi i(n_{1,2}+n_{2,3}+\cdots+n_{r-1,r})}$ denote the standard nondegenerate character of the unipotent subgroup N, where $n = (n_{i,j}) \in N$. The *boundary* Whittaker

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distribution $B = B_r$ for parameters $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$ and $\delta = (\delta_1, \ldots, \delta_r) \in (\mathbb{Z}/2\mathbb{Z})^r$ is the distribution on $GL(r, \mathbb{R})$ characterized by the transformation law

$$B(ngtn_-) \quad = \quad \psi(n)\, B(g)\, \chi_{\rho-\lambda,\delta}(t)\,, \tag{2}$$

where $g \in GL(r, \mathbb{R})$, $n \in N$, t is diagonal, $n_- \in N_-$, $\rho = (\frac{r-1}{2}, \frac{r-3}{2}, \ldots, \frac{1-r}{2})$, and

$$\chi_{\rho-\lambda,\delta}(\mathrm{diag}(t_1, \ldots, t_r)) \quad = \quad \prod_{i \le r} |t_i|^{\rho_i - \lambda_i}\, \mathrm{sgn}(t_i)^{\delta_i}.$$

This formula completely describes $B(ntn_-) = \psi(n)\chi_{\rho-\lambda,\delta}(t)$ on the open Bruhat cell of G, where it actually restricts to a function.

Consider the embedding $j : GL(r, \mathbb{R}) \hookrightarrow GL(r+1, \mathbb{R})$ into the upper left corner of $(r+1) \times (r+1)$ matrices. It has an open orbit on the product of flag varieties for these two groups. Let $f_1 \in GL(r, \mathbb{R})$ and $f_2 \in GL(r+1, \mathbb{R})$ be an arbitrary point in this orbit. The distributional archimedean integral that arises for the $GL(r) \times GL(r+1)$ Rankin-Selberg convolution is (analogously to [4])

$$\int_{B_{-,r}} B_r(b_- f_1)\, B_{r+1}(j(b_-) f_2)\, |\det b_-|^s\,, \tag{3}$$

where $B_{-,r}$ represents the lower triangular Borel subgroup of $GL(r)$. After a rational change of coordinates on $B_{-,r}$, (3) formally splits into a product of $\frac{r(r+1)}{2}$ integrals of the form (1). This gives half of the Γ-factors in the functional equation, the other half coming from the opposite side of the functional equation.

Since it is a bit lengthy to describe this coordinate change in general, we illustrate it here for some low rank cases, starting with $r = 3$. Write $b_- = \left(\begin{smallmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{smallmatrix}\right)$ and take $f_1 = I_3$, $f_2 = \left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$. For simplicity assume that $\delta = (0, 0, \ldots, 0)$ (which does not change the difficulty of the calculation). Then $B_3(b_- f_1) = |a|^{3/2-\lambda_1} |c|^{1/2-\lambda_2} |f|^{-1/2-\lambda_3}$. The factor involving B_4 is

$$B_4 \left(\begin{smallmatrix} 0 & 0 & a & a \\ 0 & c & b & b+c \\ f & e & d & d+e+f \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \quad = \quad e^{2\pi i (d+e+f)} B_4 \left(\begin{smallmatrix} 0 & 0 & a & 0 \\ 0 & c & b & 0 \\ f & e & d & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \quad = \quad e^{2\pi i (d+e+f)} B_4 \left(\begin{smallmatrix} \frac{acf}{be-cd} & -\frac{ae}{d} & a & a \\ 0 & c - \frac{be}{d} & b & b+c \\ 0 & 0 & d & d+e+f \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \tag{4}$$

by (2) (using an LU decomposition). This can be written as an explicit product involving powers of the diagonal entries, and an exponential of the sum of the ratio of each superdiagonal entry divided by the diagonal entry immediately beneath it. The successive changes of variables $b \mapsto b + cd/e, a \mapsto ab, b \mapsto bd, c \mapsto ce$ then converts the integral into a product of 6 G_δ-integrals from (1).

For $r = 4$, $f_1 = I_4$ while $f_2 = \left(\begin{smallmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right)$. Writing $b_- = \left(\begin{smallmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & i & j \end{smallmatrix}\right)$, in this case the changes of variables are $b \mapsto b + \frac{cfg - cdi}{fh - ei}$, $d \mapsto d + eg/h$, $e \mapsto e + fh/i$, followed by $a \mapsto ab$, $b \mapsto bd$, $c \mapsto ce$, $d \mapsto dg$, $e \mapsto eh$, $f \mapsto fi$.

Notice that the addition by shifts serves to simplify determinants of minors in b_- into monomials. For example, the first change of variables for $r = 4$ simplifies $\det\left(\begin{smallmatrix} b & c & 0 \\ d & e & f \\ g & h & i \end{smallmatrix}\right)$ to $b \cdot \det\left(\begin{smallmatrix} e & f \\ h & i \end{smallmatrix}\right)$, while the other shifts do something simpler for 2×2 determinants. The last phase involves multiplying each variable by the one immediately below it in the matrix, in a certain sequence. In general, the change of variables goes through the matrix in a particular order, and changes an entry in a manner in which simplifies some of the minors of the matrix. It then proceeds to change other entries, sometimes affecting ones which have already been altered.

Example 2: Exterior Square L-function on $GL(2r)$.

This example is taken from our paper [8, §4], which gives a general description of a coordinate change for matrices such as

$$\begin{pmatrix}
c_{1,1} & 0 & 0 & 0 & 0 & c_{1,1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{1,1} \\
c_{2,1} & c_{2,2} & 0 & 0 & 0 & c_{2,1}+c_{2,2} & 0 & 0 & 0 & 0 & z_{1,1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{2,2} & c_{2,1} \\
c_{3,1} & c_{3,2} & c_{3,3} & 0 & 0 & c_{3,1}+c_{3,2}+c_{3,3} & 0 & 0 & 0 & z_{2,2} & z_{2,1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{3,3} & c_{3,2} & c_{3,1} \\
c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} & 0 & c_{4,1}+c_{4,2}+c_{4,3}+c_{4,4} & 0 & 0 & z_{3,3} & z_{3,2} & z_{3,1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{4,4} & c_{4,3} & c_{4,2} & c_{4,1} \\
c_{5,1} & c_{5,2} & c_{5,3} & c_{5,4} & c_{5,5} & c_{5,1}+c_{5,2}+c_{5,3}+c_{5,4}+c_{5,5} & 0 & z_{4,4} & z_{4,3} & z_{4,2} & z_{4,1} \\
0 & 0 & 0 & 0 & 0 & 0 & c_{5,5} & c_{5,4} & c_{5,3} & c_{5,2} & c_{5,1} \\
0 & 0 & 0 & 0 & 0 & 1 & z_{5,5} & z_{5,4} & z_{5,3} & z_{5,2} & z_{5,1}
\end{pmatrix}.$$

The goal here is again to factor this matrix as ntn_- for $n \in N$, t diagonal, and $n_- \in N_-$, with an accompanying change of variables so that the superdiagonal entries in n as well as the entries of t have simple forms. This allows for the computation of Γ-factors for the exterior square L-functions.

Various shifts of variables are performed on the $z_{i,j}$ and then the $c_{i,j}$ in order to convert various minors into monomials of the variables. In this particular situation, the determinant of the bottom-right 9×9 block can be expanded by minors as $z_{1,1}\Delta_1 + (c_{2,1} + c_{2,2})\Delta_2$, where Δ_1 and Δ_2 are determinants of subblocks. We change variables $z_{1,1} \mapsto z_{1,1} - (c_{2,1} + c_{2,2})\Delta_2/\Delta_1$, so that the determinant of this 9×9 block simplifies to $z_{1,1}\Delta_1$. Similar changes of variables are done in turn for the square blocks whose bottom row is the bottom row of the matrix, and whose top right corner is $z_{2,1}, z_{2,2}, z_{3,1}, z_{3,2}, z_{3,3}, z_{4,1}, z_{4,2}, z_{4,3}$, and $z_{4,4}$ (in this order). After this is complete, similar changes of variables are then made for $c_{5,1}, c_{5,2}, c_{5,3}, c_{5,4}, c_{4,1}, c_{4,2}, c_{4,3}, c_{3,1}, c_{3,2}$, and $c_{2,1}$, in order. Note that adjusting the $c_{i,j}$'s alters the previously-changed $z_{i,j}$ in the process. The order here is different than in the Rankin-Selberg example, though ultimately for the same purpose of simplifying an integral.

Other distributional pairings give integrals which can also be calculated in terms of similar shifts. For example, Janet Chen's Ph.D. thesis [2] works out an integral on $Sp(4)$, while Brandon Bate's Ph.D. [1] thesis works out one on the exceptional group G_2.

Example 3: Existence and uniqueness of Whittaker functions.

This last example originally arose in a different application, though it shares some similarities with the previous two examples. It concerns the algebraic geometry of Schubert cells for $GL(r,\mathbb{R})$. The change of variables described below gives a desingularization of the largest Schubert cell, from which a very short proof of the existence and uniqueness of Whittaker functions (originally due to [9, 10]) can be given using our notion of a distribution *vanishing to infinite order* [5].

Consider the following ordering on the coordinates $n_{i,j}$ of matrices $n \in N$,

$$\mathcal{O}: \quad (1,2) \succ (2,3) \succ (1,3) \succ (3,4) \succ (2,4) \succ (1,4) \succ (4,5) \succ \cdots, \tag{5}$$

which is the lexicographic order on the pair $(-j,i)$ (this is not the same notion as the lexicographic ordering of a root system from Lie theory). We extend $\mathcal{O}$ in the obvious way to the positive roots α of G, corresponding to the coordinates $n_{i,j}$. Let $B_- \subset GL(r,\mathbb{R})$ be the subgroup of lower triangular matrices, and let w_{long} be the long Weyl group element, realized as the $n \times n$ identity matrix with its columns reversed.

Theorem 6. *(Miller-Schmid, 2008) There exists a birational map R*

$$\{(u_{i,j}) \,|\, 1 \le i < j \le r\} \quad \xrightarrow{R} \quad \{(n_{i,j}) \,|\, 1 \le i < j \le r\}$$

such that

(i) R is smooth, of maximal rank, on $(\mathbb{R}^*)^d$, where $d = \dim(N) = \frac{r(r-1)}{2}$

(ii) via R, the element in the $(i,i+1)$-st position in the projection of $w_{long}nB_-$ onto N modulo B_- corresponds to $\sum_{k=1}^{r-i} \frac{1}{u_{k,i+k}}$

(iii) via R, the invariant measure $\prod_{1\le i<j\le r} dn_{i,j}$ on N corresponds to $\prod_{1\le i<j\le r} u^{j-i-1} du_{i,j}$

(iv) via R, the zero sets of the functions $\prod_{i\le k<j} u_{i,j}$, $1 \le k \le r-1$, define the codimension-one Schubert cells of G.

This birational map R is defined in terms of the entries of the matrix

$$wn \quad = \quad \begin{pmatrix} & & & & 1 \\ & & & 1 & n_{r-1,r} \\ & & 1 & n_{r-2,r-1} & n_{r-2,r} \\ & \mathinner{\cdot^{\cdot^{\cdot}}} & \mathinner{\cdot^{\cdot^{\cdot}}} & \vdots & \vdots \\ 1 & n_{2,3} & \cdots & n_{2,r-1} & n_{2,r} \\ 1 \;\, n_{1,2} & n_{1,3} & \cdots & n_{1,r-1} & n_{1,r} \end{pmatrix}. \tag{7}$$

For each entry $n_\alpha = n_{i,j}$, $i < j$, in this matrix, let $\mathcal{P}_\alpha = \mathcal{P}_{i,j}$ denote the set of rectilinear paths through its entries which begin at $n_{i,r}$ and end at either $n_{j-1,j}$ or $n_{j,j+1}$, and which move only in upward and leftward steps as they pass through the matrix. For any such path p through the matrix wn, let $u(p)$ denote the product of u_γ, over all γ for which n_γ is traversed by the path. The explicit formula for the rational map is given as follows:

$$n_{i,j} \quad = \quad \frac{\sum_{p\in\mathcal{P}_{i,j}} u(p)}{u_{j,j+1}u_{j,j+2}\cdots u_{j,r}}. \tag{8}$$

For example, for $r = 3$ the matrix wn corresponds under R to

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & u_{2,3} \\ 1 & \frac{u_{1,3}u_{1,2}+u_{1,3}u_{2,3}}{u_{2,3}} & u_{1,3}\,u_{2,3} \end{pmatrix}, \tag{9}$$

while for $r = 4$ it corresponds to

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & u_{3,4} \\ 0 & 1 & \frac{u_{2,4}(u_{2,3}+u_{3,4})}{u_{3,4}} & u_{2,4}\,u_{3,4} \\ 1 & \frac{u_{1,4}(u_{1,2}u_{1,3}+u_{1,3}u_{2,3}+u_{2,3}u_{2,4})}{u_{2,3}u_{2,4}} & \frac{u_{1,4}(u_{1,3}u_{2,3}+u_{2,3}u_{2,4}+u_{2,4}u_{3,4})}{u_{3,4}} & u_{1,4}\,u_{2,4}\,u_{3,4} \end{pmatrix}. \tag{10}$$

The entries in the rightmost column always come from the sole path going straight up, meaning

$$n_{i,r} \quad = \quad \prod_{j \geq i} u_{j,r}. \tag{11}$$

This change of variables is a special case of a more general one that applies to any Schubert cell of $GL(r)$.

To conclude: **what mathematics is behind these paths and changes of variables?**

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