A problem in Euclidean Geometry

Michael Atiyah

I describe below an elementary problem in Euclidean (or Hyperbolic) geometry which remains unsolved more than 10 years after it was first formulated. There is a proof for $n = 3$ and (when the ball is the whole of 3-space) when $n = 4$. There is strong numerical evidence for $n \leqslant 30$.

Let $(x_1, x_2, ...x_n)$ be n distinct points inside the ball of radius R in Euclidean 3-space. Let the oriented line $x_i x_j$ meet the boundary 2-sphere in a point t_{ij} (regarded as a point of the complex Riemann sphere $(C \cup \infty)$). Form the complex polynomial p_i, of degree $n-1$, whose roots are t_{ij} : this is determined up to a scalar factor. The open problem is

Conjecture 1 For all $(x_1, ..., x_n)$ the n polynomials p_i are linearly independent.

Conjecture 1 is equivalent to the non-vanishing of the determinant D of the matrix of coefficients of the p_i. In fact there is a natural way of normalizing this determinant (independent of the choice of scalar factors) so that D becomes a continuous function of $(x_1, ..., x_n)$ which is $SL(2, C)$ - invariant (using the ball model of hyperbolic 3-space) Conjecture 1 can now be refined to

Conjecture 2 $|D| \geq 1$ with equality only for collinear points.

There are other versions of this conjecture, of which the most appealing and general involves 2 ellipsoids S, S' in 3-space with S inside S'. Replacing the 2-sphere above by an ellipsoid and, taking a sequence of points x_i inside S, we get two determinants D, D'. The third conjecture (which implies Conjecture 2) is

Conjecture 3 $|D'| > |D|$

More details and background can be found in the references below (but Conjecture 3 is new).

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About Poincaré Duality

by Jacob Lurie

Let M be a manifold of dimension d, and let $q : E \to M$ be a Serre fibration of topological spaces, equipped with a section $s : M \to E$. For each open subset $U \subseteq M$, let $\mathrm{Sect}_c(U)$ denote the space of maps $M \to E$ which are sections of q and which agree with s outside of a compact subset of M.

Principle 1 (Nonabelian Poincare Duality). *If the fibers of q are $(d-1)$-connected, then $\mathrm{Sect}_c(M)$ can be realized as the homotopy colimit of the diagram of spaces $\{\mathrm{Sect}_c(U)\}$, where U ranges over those open subsets of M which can be written as a finite disjoint union of disks.*

Example 2. Let A be an abelian group and let E be the product of M with an Eilenberg-MacLane space $K(A, n)$. Then we have canonical isomorphisms $\pi_i \, \mathrm{Sect}_c(M) \simeq \mathrm{H}_c^{n-i}(M; A)$. If M is an oriented manifold of dimension $d \leq n$, then the homotopy groups of the homotopy colimit of the diagram $\{\mathrm{Sect}_c(U)\}$ can be computed as the homology $\mathrm{H}_{i+d-n}(M; A)$, and Principle 1 recovers the statement of Poincare duality for the manifold M.

Example 3. Let G be a compact Lie group, let M be a compact manifold, and let E be the product of M with the classifying space of G. Then $\mathrm{Sect}_c(M)$ can be interpreted as a classifying space for G-bundles on the manifold M. Principle 1 implies that if G is simply connected and M has dimension ≤ 4 (or if G is connected and M has dimension ≤ 2), then we can reconstruct the homotopy type of this classifying space by studying G-bundles which have been trivialized outside a finite subset of M.

Problem 4 (Nonabelian Verdier Duality?). *Formulate an analogue of Principle 1 which does not require the assumption that the map $q : E \to M$ be a Serre fibration.*

DEGENERATION OF NONABELIAN HODGE STRUCTURES

CARLOS SIMPSON

The Hodge structure on the cohomology of a complex Kähler manifold has turned out to be one of the most fertile and useful structures in complex geometry. Thanks to Wilfried Schmid's work, we have a very detailed and precise understanding of how the Hodge structures degenerate when the variety becomes singular, leading to a wide array of applications in many fields.

More recently, it has appeared useful to consider the "nonabelian cohomology" of a variety, whose first basic incarnation is the moduli space of flat bundles. A natural question is to try to generalize Wilfried's structure theorems on degenerations, to the nonabelian cohomology space. This was the subject of numerous discussions with Ludmil Katzarkov and Tony Pantev in the late 1990's. Results in this direction could have applications for the study of families of varieties in diverse contexts.

Suppose $(S, 0)$ is a pointed curve and $X \to S$ is a family of curves, whose general fibers are smooth and whose special fiber X_0 is reduced with simple normal crossings. Then we can consider the moduli space $M_{DR}(X/S) \to S$ of sheaves with integrable connections on the fibers. Over a general point $s \in S$, the fiber $M_{DR}(X_s)$ parametrizes flat bundles. It has a nonabelian Hodge structure where the analogue of the Hodge metric is Hitchin's hyperkähler metric. It degenerates to a moduli space $M_{DR}(X_0)$ of torsion-free sheaves on X_0 with logarithmic connections satisfying a compatibility condition at the crossings.

Problem: Understand the degeneration of the nonabelian Hodge structures on $M_{DR}(X_s)$ as $s \to 0$. We would like to have analogues of the nilpotent and SL_2 orbit theorems, and the norm estimates. These should give asymptotic information about the degeneration of the hyperkähler metric. There should be an analogue of the Clemens-Schmid exact sequence relating flat bundles on X_0 and the residue of the nonabelian Gauss-Manin or isomonodromic deformation connection. Look for a limiting nonabelian mixed Hodge structure.

One of the difficulties is to understand what happens near points in $M_{DR}(X_0)$ parametrizing torsion-free sheaves which are not locally free at the singularities.

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Understanding restriction to K

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Suppose G is a real reductive Lie group, with maximal compact subgroup K. The representation theory of K is well-understood and well-behaved: $\widehat{K}$ is a countable discrete set consisting of finite-dimensional representations (δ, E_δ). If (π, V) is a quasisimple irreducible representation of G, Harish-Chandra proved that each irreducible representation of K appears at most finitely often in V; so there is a *multiplicity function*

$$m_V \colon \widehat{K} \to \mathbb{N}, \qquad m_V(\delta) = \dim \operatorname{Hom}_K(E_\delta, V).$$

Here is one way to study these multiplicity functions.

Theorem 1. *For every $\delta \in \widehat{K}$, there is a unique tempered irreducible representation $I(\delta)$ having real infinitesimal character, and unique lowest K-type δ. The functions $m_{I(\delta)}$ form a $\mathbb{Z}$-basis of the span of the multiplicity functions m_V. That is, for any V there is an expression*

$$m_V = \sum_{\delta \in \widehat{K}} a_V(\delta) m_{I(\delta)},$$

with $a_V(\delta) \in \mathbb{Z}$, and only finitely many $a_V(\delta)$ not equal to zero.

This is based on Schmid's results in [6]. A proof for linear G is in [7]. If δ_0 is a lowest K-type of V, then $a_V(\delta_0) = 1$. The other terms in the sum involve strictly "larger" δ, in the ordering of $\widehat{K}$ defining lowest. The Hecht-Schmid proof of Blattner's conjecture in [3] provides explicit formulas for the functions $m_{I(\delta)}$, and the Kazhdan-Lusztig conjectures allow us to calculate the integers a_V; so this theorem makes it possible to compute all of the

functions m_V. Nevertheless we do not fully *understand* these functions; the point of the problem below is to seek more geometric understanding.

Write $K(\mathbb{C})$ and $\mathfrak{g}$ for the complexifications of K and $\mathrm{Lie}(G)$. Write

$$\mathcal{N}_\theta^* = \text{cone of nilpotent elements in } (\mathfrak{g}/\mathfrak{k})^*;$$

this is an affine algebraic variety on which $K(\mathbb{C})$ acts with finitely many orbits. If $\mathcal{M}$ is a $K(\mathbb{C})$-equivariant coherent sheaf on $\mathcal{N}_\theta^*$, then the space $\Gamma\mathcal{M}$ decomposes as a direct sum of irreducible representations of K exactly as we explained above for V; so we get a multiplicity function

$$m_\mathcal{M} \colon \widehat{K} \to \mathbb{N}, \qquad m_\mathcal{M}(\delta) = \dim \mathrm{Hom}_K(E_\delta, \Gamma\mathcal{M}).$$

These multiplicity functions have a geometric character that is not evident in the representation-theoretic ones m_V. But they *include* the representation-theoretic ones.

Proposition. *Suppose V is an irreducible quasisimple representation of G. Then there is $K(\mathbb{C})$-equivariant sheaf $\mathcal{M}(V)$ on $\mathcal{N}_\theta^*$, with the property that $m_V = m_{\mathcal{M}(V)}$.*

(This equality of multiplicity functions is a consequence of a much more precise relationship between V and $\mathcal{M}(V)$, examined in detail in [8].) Here is a version of Theorem 1 for the geometric setting.

Theorem 2. *Suppose $\mathcal{O}$ is an orbit of $K(\mathbb{C})$ on $\mathcal{N}_\theta^*$, and $\mathcal{L}$ is an irreducible $K(\mathbb{C})$-equivariant vector bundle on $\mathcal{O}$. Let $\overline{\mathcal{L}}$ be any equivariant coherent sheaf on $\overline{\mathcal{O}}$ that restricts to $\mathcal{L}$. Then the multiplicity functions functions $m_{\overline{\mathcal{L}}}$ form a $\mathbb{Z}$-basis of the span of the multiplicity functions $m_\mathcal{M}$. That is, for any $K(\mathbb{C})$-equivariant coherent sheaf $\mathcal{M}$ there is an expression*

$$m_\mathcal{M} = \sum_\mathcal{L} a_\mathcal{M}(\overline{\mathcal{L}}) m_{\overline{\mathcal{L}}},$$

with $a_\mathcal{M}(\overline{\mathcal{L}}) \in \mathbb{Z}$, and only finitely many $a_\mathcal{M}(\overline{\mathcal{L}})$ not equal to zero.

This is easy. If $\mathcal{L}_0$ is a bundle supported on a component $\mathcal{O}_0$ of the support of $\mathcal{M}$, then the coefficient $a_\mathcal{M}(\overline{\mathcal{L}_0})$ is a nonnegative integer independent of all choices of coherent extensions. The other terms in the sum involve lower-dimensional orbits in the support of $\mathcal{M}$.

What has a little more content is

Proposition. *The representation-theoretic multiplicity functions m_V have exactly the same span as the geometric ones $m_\mathcal{M}$. In particular, the two bases $\{m_{I(\delta)}\}$ and $\{m_{\overline{\mathcal{L}}}\}$ are related by integer change-of-basis matrices.*

Finally we can state a problem.

Open Problem. Describe as explicitly as possible a bijection between $\widehat{K}$ and the set of irreducible $K(\mathbb{C})$-equivariant bundles $\mathcal{L}$, with the property that the change-of-basis matrix between $\{m_{I(\delta)}\}$ and $\{m_{\overline{\mathcal{L}}}\}$ is lower triangular with respect to the ordering of $\widehat{K}$ defining lowest K-types. Give an algorithm for computing this change-of-basis matrix.

For complex groups, a version of this problem was posed by Lusztig in [4], and solved by Bezrukavnikov in [2]. The version here (with a more explicit bijection) is due to Achar [1] in the case of $GL(n, \mathbb{C})$. Earlier work of Ostrik [5] is related.

For real groups, there are a number of additional difficulties. First, we did not specify how to choose $\overline{\mathcal{L}}$; making the wrong choice will interfere with lower triangularity. Second, the ordering defining lowest K-types is no longer a total order, and a single irreducible representation can have more than one lowest K-type. This difficulty partitions $\widehat{K}$ into finite subsets, each with cardinality a small power of 2 (bounded by the split rank of G). The precise desideratum is that each of these sets of representations δ should correspond to a set of the same size of bundles $\mathcal{L}$; the correspondence should make the change-of-basis matrix block lower triangular.

Computing this change of basis matrix would in particular compute the associated variety of any irreducible representation. This is an "asymptotic" description of the restriction to K, providing a useful complement to the complete and explicit formulas due to Hecht and Schmid.

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Characters and modules
James Arthur

There are two different ways to classify representations of compact connected Lie groups. One is the construction by harmonic analysis of irreducible characters. It is due to H. Weyl, in his original work that culminated in the famous Weyl character formula. (See [W, p. 377-385] for an elementary description for the case of compact unitary groups.) The other is the algebraic construction of irreducible modules by highest weights. This is best known as part of the theory of complex semisimple Lie algebras, but it is easily transformed to a classification for compact groups. The two theories each give the classification of irreducible representations in terms of their highest weights.

The problem here, which has been posed by Dihua Jiang, is to understand a similar dichotomy for automorphic representations. I am posting it because it appears to be quite natural, and because I believe that it is very important. The question also bears upon an offhand comment of Wilfried from last November,

<center>"... but what about the modules!"</center>

or words to that effect. In [A], we describe a classification of automorphic representations of orthogonal and symplectic groups. It is based on a comparison of the trace formula with its stabilization (and is still conditional on the stabilization of the twisted trace formula for GL(N), part of work in progress by Moeglin and Waldspurger). The comparison of trace formulas is a theory that rests ultimately on the characters of representations. The theta correspondence is a complementary theory based on the actual modules of representations. It has the advantage of being very explicit. The disadvantage is that it does not directly classify representations into local and global packets from which one can deduce multiplicities. In fact, it does not give an exhaustion theorem for the representations it constructs. However, initial results suggest that one might be able to have the advantages of both theories by using them together.

The problem of comparing the theta correspondence with the endoscopic classification seems to be quite complex. It might require sustained efforts from a number of mathematicians. For a more detailed description of the problem, with the initial results mentioned above, we refer the reader to [J].

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Arithmeticity (or not) of Monodromy

Peter Sarnak

April 23, 2013

In 1974 Griffiths and Schmid [1] asked whether monodromy groups of families of varieties acting on cohomology are arithmetic or not. The problem remains largely open, even for well known explicit examples. One case is that of the the families of Calabi-Yau three-folds, which have received much attention starting with the paper of Candelas-Parks *et al*. Of the well known 14 such families, 7 are known to be not arithmetic (Brav and Thomas [3]), while 3 are arithmetic (Singh -Venkataramana [5]) and 4 remain undecided. Besides deciding the remaining 4 cases the question is:

What is the geometric significance of being arithmetic?

Some further comments on this problem taken from Peter Sarnak's e-mail correspondence. Sarnak writes:

1. My interest in whether such groups are arithmetic or "thin" (the image of the monodromy group H is contained in the integer points $G(\mathbf{Z})$ of its Zariski closure. I call the group H **thin** if it is not finite index in the latter group, and arithmetic otherwise). All this is explained in my "Notes on Thin Matrix groups." While the main "expansion theorem" allows one to proceed in many cases without knowing whether the group is thin or not (and only knowing the Zariski closure—just as with most arithmetic geometric applications), in the diophantine orbit world one does want to know more. E.g if the group happens to be arithmetic then one can often use automorphic forms, Galois cohomology and other methods to resolve a problem. These are much more powerful than the substitute theory for "thin" groups.

2. A second reason to be interested in this is curiosity. That is, can one really compute a monodromy group? The first question after the Zariski closure is whether it is thin or not. The general question of whether the group is thin or not has no decision procedure. The situation is very similar to Hilbert's 10-th problem for decision procedures for diophantine problems. That is the local obstructions are the finite quotients of the (say, monodromy) group H. These can be identical to a determinable congruence subgroup $K \subset G(\mathbf{Z})$ even when H is of infinite index in $G(\mathbf{Z})$. So in this case one passes all local tests; so now, how to tell if H is this unique congruence subgroup K or is thin? If it is K, one might certify it is so by exhibiting the generators of K as products of the generators of H (which is, in fact, then manner in which H is given). The analogue of this in Hilbert 11 is that one gives an integral point on the variety demonstrating it has a point. The problematic step is giving

a certificate that H is thin. One method (which I view as the analogue of the method of descent) is to show that the generators of H play ping pong on some set. This means that there is a way of seeing expressions in the generators are getting more complex and hence of understanding the structure of H with its generators combinatorially. This can often be combined with cohomological methods to give a certificate that a group is thin. The trouble with this is it hard to show that the generators do, in fact, play ping pong. In some examples this can be done and this is what Brav and Thomas did for the Dwork $n = 4$ case. My talk will be about a certificate for being thin, which is a bit like the Brauer-Manin obstruction. It applies to the hypergeometric monodromy groups $n\_F_{(}n-1)$ when their Zariski closure is orthogonal of signature $(n-1, 1)$ and it is quite robust.

3. I don't have any good ideas about the geometric significance of being thin or not (and I would love to hear some ideas). However my feeling is that being thin is exotic enough that in some examples it is the reason it carries precious information (somehow if the the group were arithmetic the information gleaned would have been extracted by other means). To back this up note that the Dwork Family used by Taylor et al (with n equal to 4 and higher) is I expect thin. For $n = 4$ this is proved by Brav-Thomas in [3]. Also the Candelas case—which set off mirror symmetry story—is thin. Kontsevich has some ideas coming from dynamics connected to variation of Hodge structures that might explain the significance of thin. See the report on his recent lecture [4].

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Feynman Diagrams

by Matvei Libine (joint with with Igor Frenkel)

Feynman diagrams are a pictorial way of describing integrals predicting possible outcomes of interactions of subatomic particles in the context of quantum field physics. As the number of variables that are being integrated out increases, the integrals become more and more difficult to compute. But in the cases when the integrals can be computed, the accuracy of their prediction is amazing. Many of these diagrams corresponding to real-world scenarios result in integrals that are divergent in mathematical sense. Physicists have a collection of competing techniques called "renormalization" of Feynman integrals which "cancel out the infinities" coming from different parts of the diagrams. After renormalization, calculations using Feynman diagrams match experimental results with very high accuracy. However, these renormalization techniques appear very suspicious to mathematicians and attract criticism from physicists as well. For example, do you get the same result if you apply a different technique? If the results are different, how do you choose the "right" technique? Or, if the results are the same, what is the reason for that? Most of these questions will be resolved if one finds an intrinsic mathematical meaning of Feynman diagrams, most likely as projectors onto irreducible components of certain representations. A number of mathematicians already work on this problem of finding a mathematical interpretation of Feynman diagrams, mostly in the setting of algebraic geometry. Recently published book "Feynman Motives" by M. Marcolli, [1] contains a summary of these algebraic-geometric developments as well as a comprehensive list of references. However, there is a strong evidence that at least some Feynman diagrams should have a representation-theoretic interpretation. The answer might be as simple and elegant as projectors onto irreducible components of certain representations of $U(2,2)$.

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What mathematics is really behind the distributional Γ-factors?

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This problem is motivated by my joint work with Wilfried Schmid on constructing L-functions via automorphic distributions (see [6–8]). An interesting – and poorly understood – aspect of our method is the structure of its archimedean integrals. In every case examined thus far, there is some change of coordinates that splits them into products of integrals of the form

$$
G_\delta(s) \quad := \quad \int_{\mathbb{R}} e^{2\pi i x} \operatorname{sgn}(x)^\delta \, dx \quad = \quad i^\delta \, \frac{\Gamma_{\mathbb{R}}(s+\delta)}{\Gamma_{\mathbb{R}}(1-s+\delta)} \,, \quad \delta \in \{0,1\} \,, \tag{1}
$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$ is the factor that famously accompanies $\zeta(s)$ in its functional equation. This identity is first proved by a contour shift when $0 < \operatorname{Re} s < 1$ (where it conditionally converges), and then extends to $s \in \mathbb{C}$ by meromorphic continuation.

Despite the uniformity of the answers we obtain, the computations have been performed by *ad hoc* combinatorial methods. I'd like to describe some examples here in the hopes that an appropriate algebraic context can be found to explain them. For that reason all integrals below will be expressed formally, without concern for convergence.

The first two examples are related to specific Γ-factor computations, while the last arose in understanding the existence and uniqueness of Whittaker functions on the group $GL(r, \mathbb{R})$. Both have vague resemblances to formulas from cluster algebras in [3]. What is really going on behind this, making it work?

Example 1: Rankin-Selberg tensor product on $GL(r) \times GL(r+1)$.

Let N and N_- be the subgroups of $r \times r$ unipotent upper and lower triangular matrices in $GL(r, \mathbb{R})$, respectively. Let $\psi(n) = e^{2\pi i(n_{1,2}+n_{2,3}+\cdots+n_{r-1,r})}$ denote the standard nondegenerate character of the unipotent subgroup N, where $n = (n_{i,j}) \in N$. The *boundary* Whittaker

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distribution $B = B_r$ for parameters $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$ and $\delta = (\delta_1, \ldots, \delta_r) \in (\mathbb{Z}/2\mathbb{Z})^r$ is the distribution on $GL(r, \mathbb{R})$ characterized by the transformation law

$$B(ngtn_-) = \psi(n) B(g) \chi_{\rho - \lambda, \delta}(t), \tag{2}$$

where $g \in GL(r, \mathbb{R})$, $n \in N$, t is diagonal, $n_- \in N_-$, $\rho = (\frac{r-1}{2}, \frac{r-3}{2}, \ldots, \frac{1-r}{2})$, and

$$\chi_{\rho - \lambda, \delta}(\mathrm{diag}(t_1, \ldots, t_r)) = \prod_{i \leq r} |t_i|^{\rho_i - \lambda_i} \mathrm{sgn}(t_i)^{\delta_i}.$$

This formula completely describes $B(ntn_-) = \psi(n)\chi_{\rho - \lambda, \delta}(t)$ on the open Bruhat cell of G, where it actually restricts to a function.

Consider the embedding $j : GL(r, \mathbb{R}) \hookrightarrow GL(r+1, \mathbb{R})$ into the upper left corner of $(r+1) \times (r+1)$ matrices. It has an open orbit on the product of flag varieties for these two groups. Let $f_1 \in GL(r, \mathbb{R})$ and $f_2 \in GL(r+1, \mathbb{R})$ be an arbitrary point in this orbit. The distributional archimedean integral that arises for the $GL(r) \times GL(r+1)$ Rankin-Selberg convolution is (analogously to [4])

$$\int_{B_{-,r}} B_r(b_- f_1) B_{r+1}(j(b_-) f_2) \, |\det b_-|^s, \tag{3}$$

where $B_{-,r}$ represents the lower triangular Borel subgroup of $GL(r)$. After a rational change of coordinates on $B_{-,r}$, (3) formally splits into a product of $\frac{r(r+1)}{2}$ integrals of the form (1). This gives half of the Γ-factors in the functional equation, the other half coming from the opposite side of the functional equation.

Since it is a bit lengthy to describe this coordinate change in general, we illustrate it here for some low rank cases, starting with $r = 3$. Write $b_- = \left(\begin{smallmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{smallmatrix}\right)$ and take $f_1 = I_3$, $f_2 = \left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$. For simplicity assume that $\delta = (0, 0, \ldots, 0)$ (which does not change the difficulty of the calculation). Then $B_3(b_- f_1) = |a|^{3/2 - \lambda_1} |c|^{1/2 - \lambda_2} |f|^{-1/2 - \lambda_3}$. The factor involving B_4 is

$$B_4 \left(\begin{smallmatrix} 0 & 0 & a & a \\ 0 & c & b & b+c \\ f & e & d & d+e+f \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) = e^{2\pi i(d+e+f)} B_4 \left(\begin{smallmatrix} 0 & 0 & a & 0 \\ 0 & c & b & 0 \\ f & e & d & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) = e^{2\pi i(d+e+f)} B_4 \left(\begin{smallmatrix} \frac{acf}{be-cd} & -\frac{ae}{d} & a & a \\ 0 & c-\frac{be}{d} & b & b+c \\ 0 & 0 & d & d+e+f \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \tag{4}$$

by (2) (using an LU decomposition). This can be written as an explicit product involving powers of the diagonal entries, and an exponential of the sum of the ratio of each superdiagonal entry divided by the diagonal entry immediately beneath it. The successive changes of variables $b \mapsto b + cd/e, a \mapsto ab, b \mapsto bd, c \mapsto ce$ then converts the integral into a product of 6 G_δ-integrals from (1).

For $r = 4$, $f_1 = I_4$ while $f_2 = \left(\begin{smallmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right)$. Writing $b_- = \left(\begin{smallmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & i & j \end{smallmatrix}\right)$, in this case the changes of variables are $b \mapsto b + \frac{cfg - cdi}{fh - ei}$, $d \mapsto d + eg/h$, $e \mapsto e + fh/i$, followed by $a \mapsto ab$, $b \mapsto bd$, $c \mapsto ce$, $d \mapsto dg$, $e \mapsto eh$, $f \mapsto fi$.

Notice that the addition by shifts serves to simplify determinants of minors in b_- into monomials. For example, the first change of variables for $r = 4$ simplifies $\det \left(\begin{smallmatrix} b & c & 0 \\ d & e & f \\ g & h & i \end{smallmatrix} \right)$ to $b \cdot \det \left(\begin{smallmatrix} e & f \\ h & i \end{smallmatrix} \right)$, while the other shifts do something simpler for 2×2 determinants. The last phase involves multiplying each variable by the one immediately below it in the matrix, in a certain sequence. In general, the change of variables goes through the matrix in a particular order, and changes an entry in a manner in which simplifies some of the minors of the matrix. It then proceeds to change other entries, sometimes affecting ones which have already been altered.

Example 2: Exterior Square L-function on $GL(2r)$.

This example is taken from our paper [8, §4], which gives a general description of a coordinate change for matrices such as

$$\left(\begin{smallmatrix}
c_{1,1} & 0 & 0 & 0 & 0 & c_{1,1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{1,1} \\
c_{2,1} & c_{2,2} & 0 & 0 & 0 & c_{2,1}+c_{2,2} & 0 & 0 & 0 & 0 & z_{1,1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{2,2} & c_{2,1} \\
c_{3,1} & c_{3,2} & c_{3,3} & 0 & 0 & c_{3,1}+c_{3,2}+c_{3,3} & 0 & 0 & 0 & z_{2,2} & z_{2,1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{3,3} & c_{3,2} & c_{3,1} \\
c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} & 0 & c_{4,1}+c_{4,2}+c_{4,3}+c_{4,4} & 0 & 0 & z_{3,3} & z_{3,2} & z_{3,1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{4,4} & c_{4,3} & c_{4,2} & c_{4,1} \\
c_{5,1} & c_{5,2} & c_{5,3} & c_{5,4} & c_{5,5} & c_{5,1}+c_{5,2}+c_{5,3}+c_{5,4}+c_{5,5} & 0 & z_{4,4} & z_{4,3} & z_{4,2} & z_{4,1} \\
0 & 0 & 0 & 0 & 0 & 0 & c_{5,5} & c_{5,4} & c_{5,3} & c_{5,2} & c_{5,1} \\
0 & 0 & 0 & 0 & 0 & 1 & z_{5,5} & z_{5,4} & z_{5,3} & z_{5,2} & z_{5,1}
\end{smallmatrix} \right).$$

The goal here is again to factor this matrix as ntn_- for $n \in N$, t diagonal, and $n_- \in N_-$, with an accompanying change of variables so that the superdiagonal entries in n as well as the entries of t have simple forms. This allows for the computation of Γ-factors for the exterior square L-functions.

Various shifts of variables are performed on the $z_{i,j}$ and then the $c_{i,j}$ in order to convert various minors into monomials of the variables. In this particular situation, the determinant of the bottom-right 9×9 block can be expanded by minors as $z_{1,1}\Delta_1 + (c_{2,1} + c_{2,2})\Delta_2$, where Δ_1 and Δ_2 are determinants of subblocks. We change variables $z_{1,1} \mapsto z_{1,1} - (c_{2,1} + c_{2,2})\Delta_2/\Delta_1$, so that the determinant of this 9×9 block simplifies to $z_{1,1}\Delta_1$. Similar changes of variables are done in turn for the square blocks whose bottom row is the bottom row of the matrix, and whose top right corner is $z_{2,1}, z_{2,2}, z_{3,1}, z_{3,2}, z_{3,3}, z_{4,1}, z_{4,2}, z_{4,3}$, and $z_{4,4}$ (in this order). After this is complete, similar changes of variables are then made for $c_{5,1}, c_{5,2}, c_{5,3}, c_{5,4}, c_{4,1}, c_{4,2}, c_{4,3}, c_{3,1}, c_{3,2}$, and $c_{2,1}$, in order. Note that adjusting the $c_{i,j}$'s alters the previously-changed $z_{i,j}$ in the process. The order here is different than in the Rankin-Selberg example, though ultimately for the same purpose of simplifying an integral.

Other distributional pairings give integrals which can also be calculated in terms of similar shifts. For example, Janet Chen's Ph.D. thesis [2] works out an integral on $Sp(4)$, while Brandon Bate's Ph.D. [1] thesis works out one on the exceptional group G_2.

Example 3: Existence and uniqueness of Whittaker functions.

This last example originally arose in a different application, though it shares some similarities with the previous two examples. It concerns the algebraic geometry of Schubert cells for $GL(r, \mathbb{R})$. The change of variables described below gives a desingularization of the largest Schubert cell, from which a very short proof of the existence and uniqueness of Whittaker functions (originally due to [9,10]) can be given using our notion of a distribution *vanishing to infinite order* [5].

Consider the following ordering on the coordinates $n_{i,j}$ of matrices $n \in N$,

$$\mathcal{O}: \quad (1,2) \succ (2,3) \succ (1,3) \succ (3,4) \succ (2,4) \succ (1,4) \succ (4,5) \succ \cdots, \quad (5)$$

which is the lexicographic order on the pair $(-j, i)$ (this is not the same notion as the lexicographic ordering of a root system from Lie theory). We extend $\mathcal{O}$ in the obvious way to the positive roots α of G, corresponding to the coordinates $n_{i,j}$. Let $B_- \subset GL(r, \mathbb{R})$ be the subgroup of lower triangular matrices, and let w_{long} be the long Weyl group element, realized as the $n \times n$ identity matrix with its columns reversed.

Theorem 6. *(Miller-Schmid, 2008) There exists a birational map R*

$$\{(u_{i,j}) \,|\, 1 \le i < j \le r\} \quad \xrightarrow{R} \quad \{(n_{i,j}) \,|\, 1 \le i < j \le r\}$$

such that

(i) *R is smooth, of maximal rank, on $(\mathbb{R}^*)^d$, where $d = \dim(N) = \frac{r(r-1)}{2}$*

(ii) *via R, the element in the $(i, i+1)$-st position in the projection of $w_{long} n B_-$ onto N modulo B_- corresponds to $\sum_{k=1}^{r-i} \frac{1}{u_{k,i+k}}$*

(iii) *via R, the invariant measure $\prod_{1 \le i < j \le r} dn_{i,j}$ on N corresponds to $\prod_{1 \le i < j \le r} u^{j-i-1} du_{i,j}$*

(iv) *via R, the zero sets of the functions $\prod_{i \le k < j} u_{i,j}$, $1 \le k \le r-1$, define the codimension-one Schubert cells of G.*

This birational map R is defined in terms of the entries of the matrix

$$wn \;=\; \begin{pmatrix} & & & & 1 \\ & & & 1 & n_{r-1,r} \\ & & 1 & n_{r-2,r-1} & n_{r-2,r} \\ & \iddots & \iddots & \vdots & \vdots \\ 1 & n_{2,3} & \cdots & n_{2,r-1} & n_{2,r} \\ 1 & n_{1,2} & n_{1,3} & \cdots & n_{1,r-1} & n_{1,r} \end{pmatrix}. \quad (7)$$

For each entry $n_\alpha = n_{i,j}$, $i < j$, in this matrix, let $\mathcal{P}_\alpha = \mathcal{P}_{i,j}$ denote the set of rectilinear paths through its entries which begin at $n_{i,r}$ and end at either $n_{j-1,j}$ or $n_{j,j+1}$, and which move only in upward and leftward steps as they pass through the matrix. For any such path p through the matrix wn, let $u(p)$ denote the product of u_γ, over all γ for which n_γ is traversed by the path. The explicit formula for the rational map is given as follows:

$$n_{i,j} \;=\; \frac{\sum_{p \in \mathcal{P}_{i,j}} u(p)}{u_{j,j+1} u_{j,j+2} \cdots u_{j,r}}. \quad (8)$$

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For example, for $r = 3$ the matrix wn corresponds under R to

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & u_{2,3} \\ 1 & \frac{u_{1,3}u_{1,2}+u_{1,3}u_{2,3}}{u_{2,3}} & u_{1,3}\,u_{2,3} \end{pmatrix}, \tag{9}$$

while for $r = 4$ it corresponds to

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & u_{3,4} \\ 0 & 1 & \frac{u_{2,4}(u_{2,3}+u_{3,4})}{u_{3,4}} & u_{2,4}\,u_{3,4} \\ 1 & \frac{u_{1,4}(u_{1,2}u_{1,3}+u_{1,3}u_{2,3}+u_{2,3}u_{2,4})}{u_{2,3}u_{2,4}} & \frac{u_{1,4}(u_{1,3}u_{2,3}+u_{2,3}u_{2,4}+u_{2,4}u_{3,4})}{u_{3,4}} & u_{1,4}\,u_{2,4}\,u_{3,4} \end{pmatrix}. \tag{10}$$

The entries in the rightmost column always come from the sole path going straight up, meaning

$$n_{i,r} \;=\; \prod_{j \geq i} u_{j,r}. \tag{11}$$

This change of variables is a special case of a more general one that applies to any Schubert cell of $GL(r)$.

To conclude: **what mathematics is behind these paths and changes of variables?**

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Tautological classes on the moduli space of $K3$ surfaces

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We denote by $\mathcal{K}_\ell$ the moduli stack of quasipolarized $K3$ surfaces (X, H) of degree $H^2 = 2\ell$, and let

$$\pi : \mathcal{X} \to \mathcal{K}_\ell$$

be the universal surface, equipped with the universal quasipolarization $\mathcal{H} \to \mathcal{X}$.

The Hodge line bundle

$$V = \left(R^2\pi_\star\mathcal{O}_\mathcal{X}\right)^{-1}$$

gives rise to a natural divisor class

$$\lambda = c_1(V),$$

generating a subring $\langle\lambda\rangle \subset A^\star(\mathcal{K}_\ell)$ in the Chow of $\mathcal{K}_\ell$. In [GK], the authors consider the Chern classes $c_1 = \pi^\star\lambda$ and c_2 of the relative cotangent bundle $\Omega^1_{\mathcal{X}/\mathcal{K}_\ell}$, and calculate that

$$\pi_\star c_2^m \in \langle\lambda\rangle, \text{ for all } m.$$

Beyond the universal surface $\mathcal{X}$, we may contemplate more general geometric structures over $\mathcal{K}_\ell$, and could ask: do they give rise to new natural classes in the Chow ring of $\mathcal{K}_\ell$ or does the λ-ring entirely capture the tautological cycle structure of $\mathcal{K}_\ell$?

As a concrete example, for a fixed integer n, we consider the relative Hilbert scheme of n points

$$\pi : \mathcal{X}^{[n]} \to \mathcal{K}_\ell.$$

(For simplicity we let π denote the projection to $\mathcal{K}_\ell$ in all considered contexts.) We let $\mathbb{D} \subset \mathcal{X}^{[n]}$ be the natural diagonal divisor of subschemes whose support points are not all distinct. In other words, fiberwise over a quasipolarized (X, H), $\mathbb{D}$ consists of the length n zero-dimensional subschemes of X supported at at most $n-1$ distinct points of X. We let $\delta \in A^1(\mathcal{X}^{[n]})$ be the corresponding Chow class, and ask

Question 1. *Are the pushforwards $\pi_\star\delta^m$ for $m > 2n$ in the λ-ring?*

The Hilbert scheme can be viewed as the relative moduli stack of rank 1 torsion free sheaves of trivial determinant and second Chern number $-n$. More generally, it is natural to consider spaces of higher rank sheaves on a $K3$, as the surface varies in moduli. We restrict attention to the open substack $\mathcal{K}_\ell^\circ \subset \mathcal{K}_\ell$ where the line bundle $\mathcal{H}$ over the universal surface is ample, and construct

$$M[v] \to \mathcal{K}_\ell^\circ,$$

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the moduli space of $\mathcal{H}$-semistable sheaves with rank r, determinant $d\mathcal{H}$ and Euler characteristic $a - r$ over the fibers of $\pi : \mathcal{X}^\circ \to \mathcal{K}_\ell^\circ$. Over a fixed polarized $K3$ surface (X, H), the moduli space consists of sheaves F with Mukai vector

$$v(F) = \mathrm{ch} F \sqrt{\mathrm{todd}\, X} = r + dH + a[\mathrm{pt}] \in H^\star(X, \mathbb{Z}).$$

We may consider an additional Mukai vector $w = s + eH + b[\mathrm{pt}] \in H^\star(X, \mathbb{Z})$, complementary in the sense that

$$\chi(v \cdot w) = 0 \ \text{ on } \ X.$$

We form the second relative moduli space $M[w] \to \mathcal{K}_\ell^\circ$, and note that the product

$$\pi : M[v] \times_{\mathcal{K}_\ell^\circ} M[w] \to \mathcal{K}_\ell^\circ$$

comes endowed with a canonical Brill-Noether locus

(1) $$\{(X, H, E, F) \ \text{ so that } \ \mathbb{H}^0(X, E \otimes^{\mathbf{L}} F) \neq 0\} \subset M[v] \times_{\mathcal{K}_\ell^\circ} M[w],$$

which is expected divisorial. The corresponding line bundle $\Theta \to M[v] \times_{\mathcal{K}_\ell^\circ} M[w]$ is in any case always defined. We ask

Question 2. *Is the Chern character* $\mathrm{ch}\,(\mathbf{R}\pi_\star\Theta)$ *in the ring generated by the Hodge class* $\lambda = -c_1(R^2\pi_\star\mathcal{O}_{\mathcal{X}^\circ})$ *?*

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