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MATHEMATICS AS AN ADEQUATE LANGUAGE

(a few remarks)

PLAN

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INTRODUCTION

This conference is called “The unity of mathematics”. I would like to make a few remarks on this wonderful theme.

I do not consider myself a prophet. I am simply a student. All my life I have been learning from great mathematicians such as Euler and Gauss, from my older and younger colleagues, from my friends and collaborators, and most importantly from my students. This is my way to continue working.

Many people consider mathematics to be a boring and formal science. However, any really good work in mathematics always has in it: beauty, simplicity, exactness, and crazy ideas. This is a strange combination. I understood earlier that this combination is essential on the example of classical music and poetry. But it is also typical in mathematics. It is not by chance that many mathematicians enjoy serious music.

This combination of beauty, simplicity, exactness and crazy ideas is, I think, common to both mathematics and music. When we think about music we do not divine it into specific areas as we often do in mathematics. If we ask a composer what is his profession, he will answer: “I am a composer.” He is unlikely to answer, ”I am a composer of quartets.” Maybe this is the reason why when I am asked what kind of mathematics I do, I just answer, “I am a mathematician”.

I was lucky to meet the great Paul Dirac, with whom I spent a few days in Hungary. I learned a lot from him.

In the 1930’s, a young physicist, Pauli, wrote one of the best books on quantum mechanics. In the last chapter of this book, Pauli discusses the Dirac equations. He writes that Dirac equations have weak points because they yield improbable and even crazy conclusions:

1. These equations assume that, besides an electron, there exists a positively charged particle, the positron, which no one ever observed.

2. Moreover, the electron behaves strangely upon meeting the positron. The two annihilate each other and form two protons.

And what is completely crazy:

3. Two photons can turn into an electron-positron pair.

Pauli writes that dispite this, the Dirac equations are quite interesting and especially the Dirac matrices deserve attention.

I asked Dirac,

“Paul, why, in spite of these comments, did you not abandon your equations and continue to pursue your results?”

“Because, they are beautiful.”

Now it is time for a radical perestroika of the fundamental language of mathematics. I will talk about this later. During this time, it is especially important to remember the unity of mathematics, to remember its beauty, simplicity, exactness and crazy ideas.

It is very useful for me to remind myself than when the style of music changed in the 20th century many people said that the modern music lacked harmony, did not follow standard rules, had dissonances, and so on. However, Shoenberg, Stravinsky, Shostakovich and Schnitke were as exact in their music as Bach, Mozart and Beethoven.

1. NONCOMMUTATIVE MULTIPLICATION

We may start with rethinking relations between two simplest operations: addition and multiplication.

The traditional Arithmetic and Algebra are too restrictive. They originate from a simple counting and they describe and canonize simplest relations between persons, groups, cells, etc. This language is sequential: to perform operations is like reading a book, and the axiomatic of this language (rings, algebras, skew-fields, categories) is too rigid. For example, a theorem by Wedderburn states that a finite-dimensional division algebra is always commutative.

1.1. Noncommutative high-school algebra.

For twelve years V. Retakh and I tried to understand associative noncommutative multiplication. This is the simplest possible operation: you operate with words in a given alphabet without any brackets and you multiply the words by concatenation. Part of these results are described in a recent survey "Quasideterminants" by I. Gelfand, S. Gelfand, V. Retakh and R. Wilson. I would say that noncommutative mathematics is as simple (or, even more simple) than the commutative one, but it is different. It is surprising how rich this structure is.

Take a quadratic equation

$$x^2 + px + q = 0$$

over a division algebra. Let x_1, x_2 be its left roots, i.e. $x_i^2 + px_i + q = 0$, $i = 1, 2$. You cannot write $-p = x_1 + x_2$, $q = x_1 x_2$ as in the commutative case. To have the proper formulas we have to give other clothes to x_1 and x_2. Namely, assume that the difference is invertible and set $x_{2,1} = (x_1 - x_2)x_1(x_1 - x_2)^{-1}$, $x_{1,2} = (x_2 - x_1)x_2(x_2 - x_1)^{-1}$. Then

$$-p = x_{1,2} + x_1 = x_{2,1} + x_2,$$

$$q = x_{1,2}x_1 = x_{2,1}x_2.$$

To generalize this theorem to polynomials of the n-th degree with left roots $x_1, \ldots, x_n$ we need to find "new clothes" for these roots by following the same pattern. For any subset $A \subset \{1, \ldots, n\}$, $A = (i_1, \ldots, i_m)$ and $i \notin A$ we introduce *pseudo-roots* $x_{A,i}$. They are given by the formula

$$x_{A,i} = v(x_{i_1}, \ldots, x_{i_m}, x_i)x_i v(x_{i_1}, \ldots, x_{i_m}, x_i)^{-1},$$

where $v(x_{i_1}, \dots, x_{i_m}, x_i)$ is the Vandermonde quasideterminant, $v(x_i) = 1$,

$$v(x_{i_1}, \dots, x_{i_m}, x_i) = \begin{vmatrix} x_{i_1}^m & \dots & x_{i_m}^m & \boxed{x_i^m} \\ & \dots & & \\ x_{i_1} & \dots & x_{i_m} & x_i \\ 1 & \dots & 1 & 1 \end{vmatrix}.$$

Suppose now that roots $x_1, \dots, x_n$ are multiplicity free, i.e. the differences $x_{A,i} - x_{A,j}$ are invertible for any A and $i \notin A$, $j \notin A$, $i \neq j$.

Let $x_1, \dots, x_n$ be multiplicity free roots of the equation

$$x^n + a_1 x^{n-1} + \dots + a_n = 0.$$

Let $(i_1, \dots, i_n)$ be an ordering of $1, \dots, n$. Set $\tilde{x}_{i_k} = x_{\{i_1,\dots,i_{k-1}\},i_k}$, $k = 1, \dots, n$.

Theorem.

$$-a_1 = \tilde{x}_{i_n} + \dots + \tilde{x}_{i_1},$$

$$a_2 = \sum_{p>q} \tilde{x}_{i_p} \tilde{x}_{i_q},$$

$$\dots,$$

$$a_n = (-1)^n \tilde{x}_{i_n} \dots \tilde{x}_{i_1}.$$

These formulas lead to a factorization

$$P(t) = (t - \tilde{x}_{i_n})(t - \tilde{x}_{i_{n-1}}) \dots (t - \tilde{x}_{i_1}),$$

where $P(t) = t^n + a_1 t^{n-1} + \dots + a_n$ and t is a central variable.

Thus, if the roots are multiplicity free, then we have $n!$ different factorizations of $P(t)$. In the commutative case we also have $n!$ factorizations of $P(t)$ but they all coincide.

Variables $x_{A,i}$ satisfy relations

$$x_{A\cup\{i\},j} + x_{A,i} = x_{A\cup\{j\},i} + x_{A,j},$$

$$x_{A\cup\{i\},j} x_{A,i} = x_{A\cup\{j\},i} x_{A,j}$$

for $i \notin A$, $j \notin A$.

The algebra generated by these variables and these relations is called Q_n. This is a universal algebra of pseudo-roots of noncommutative polynomials. By going to quotients of this algebra, we may study special polynomials, for example, polynomials with multiple roots when $x_{A,i} = x_{A,j}$ for some i, j and A. Even to a trivial polynomial x^n there corresponds an interesting quotient algebra Q_n^0 of Q_n. For example, Q_2^0 is a nontrivial algebra with generators x_1, x_2 and relations $x_1^2 = x_2^2 = 0$.

Note that, Q_n is a Koszul (i.e. "good") algebra and its dual also has an interesting structure.

1.2. Algebras with two multiplications.

Sometimes a simple multiplication is a sum of two even simpler multiplications. A good example is the algebra of noncommutative symmetric functions studied by Retakh, R. Wilson and me. In notations of Section 1.1, this algebra can be described as follows. Let $x_1, \dots, x_n$ be free noncommuting variables. Let $i_1, \dots, i_n$ be an ordering of $1, \dots, n$. Define elements $\tilde{x}_{i_1}, \dots, \tilde{x}_{i_n}$ as above. Let Sym be the algebra of polynomials in $\tilde{x}_{i_1}, \dots, \tilde{x}_{i_n}$ which are symmetric in $x_1, \dots, x_n$ as rational functions. Algebra Sym does not depend on an ordering of $1, \dots, n$, and we call it the algebra of noncommutative symmetric functions in variables $x_1, \dots, x_n$.

To construct a linear basis in algebra Sym, we need some notations. Let $w = a_{p_1} \dots a_{p_k}$ be a word in ordered letters $a_1 < \dots < a_n$. An integer m is called a *descent* of w if $m < k$ and $p_m > p_{m+1}$. Let $M(w)$ be the set of all descents of w.

Choose any ordering of $x_1, \dots, x_n$, say, $x_1 < x_2 < \dots < x_n$. For any set $J = (j_1, \dots, j_k)$ define

$$R_J = \sum \tilde{x}_{p_1} \dots \tilde{x}_{p_m},$$

where the sum is taken over all words $w = x_{p_1} \dots x_{p_m}$ such that $M(w) = \{j_1, j_1 + j_2, \dots, j_1 + j_2 + \dots + j_{k-1}\}$.

Polynomials R_J are called ribbon Schur functions, they are noncommutative analogs of commutative ribbon Schur functions introduced by MacMahon.

One can define two multiplications on noncommutative ribbon Schur functions. Let $I = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_s)$. Set $I + J = (i_1, \dots, i_{r-1}, i_r + j_1, j_2, \dots, j_s)$, $I \cdot J = (i_1, \dots, i_{r-1}, i_r, j_1, j_2, \dots, j_s)$.

Set

$$R_I *_1 R_J = R_{I+J}, \quad R_I *_2 R_J = R_{I.J}.$$

Multiplications $*_1$ and $*_2$ are associative and their sum equals the standard multiplication in Sym. In other words,

$$R_I R_J = R_{I+J} + R_{I.J}.$$

In fact, algebra Sym is freely generated by one element $\tilde{x}_1 + \cdots + \tilde{x}_n$ and two multiplications $*_1$ and $*_2$.

Two multiplications also play a fundamental role in the theory of integrable systems of Magri–Dorfman–Gelfand–Zakharevich. The theory is based on a pair of Poisson brackets such that any their linear combination is a Poisson bracket. The Kontsevich quantization of this structure gives us a family of associative multiplications.

I think it is time to study several multiplications. It may bring a lot of new connections.

1.3. Heredity vs multiplicativity.

An important problem both in pure and applied mathematics is how to deal with block-matrices. Attempts to find an adequate language for this problem go back to Frobenius and Schur. My colleagues and I think that we found an adequate language: quasideterminants. Quasideterminants do not possess the multiplicative property of determinants but unlike commutative determinants they satisfy the more important "Heredity Principle": let A be a square matrix over a division algebra and (A_{ij}) a block decomposition of A. Consider A_{ij}'s as elements of a matrix X. Then the quasideterminant of X will be a matrix B, and (under natural assumptions) the quasideterminant of B is equal to a suitable quasideterminant of A. Maybe, instead of categories one should study structures with the "Heredity Principle."

The determinants of multi-dimensional matrices also do not satisfy the multiplicative property. One cannot be too traditional here and to be restrained by requiring the multiplicative property of determinants. I think we have found an adequate language for dealing with multi-dimensional matrices (see the book " Discriminants, Resultants and Multidimensional Determinants" by I.Gelfand, M. Kapranov, and A. Zelevinsky). A beautiful application of this technique connecting Multilinear Algebra and Classical Number theory was given in the dissertation "Higher composition laws" by M. Bhargava. I may predict that this is just a beginning.

2. ADDITION AND MULTIPLICATION

The simplicity of the relations between addition and multiplication is sometimes illusory. A free abelian group with one generator (denoted 1) and with operation of addition and a free abelian monoid with infinitely many generators and with operation of multiplication (called prime numbers) are the simplest objects one can imagine, but their "marriage" gives us the ring of integers Z.

And even Gross, Iwaniec, and Sarnak cannot answer all questions about the mysteries of the ring of integers. To solve the Riemann hypothesis, for example.

The great physicist Lev Landau noticed: "I do not understand why mathematicians try to prove theorems about addition of prime numbers. Prime numbers were invented to multiply them and not to add." But for a mathematician, the nature of addition of prime numbers is a key point in understanding the relations between two operations: addition and multiplication.

Note, that theories like Minkowski mixed volumes and valuations are very interesting forms of addition.

Invention of different types of canonical bases (Gelfand–Zetlin, Kazhdan–Lusztig, Lusztig, Kashiwara, Berenstein–Zelevinsky) are, in facts, attempts to relate addition and multiplication. Many good bases have a geometric nature: they are related or they should be related with triangulations of some polyhedra.

Another attempt is the invention of matroids by Whitney. Whitney tried to axiomatize a notion of linear independence for vectors. This gives interesting connections between Algebra and Combinatorial Geometry. I will talk about this later.

Algebraic aspects of different types of matroids, including Coxeter matroids introduced by Serganova and me, are discussed in a recent book "Coxeter matroids" by A. Borovik, I. Gelfand, and N. White. But this is just a beginning. In particular, we have to invent matroids in Noncommutative Algebra and Geometry.

3. GEOMETRY

Geometry has a different nature compared to Algebra: it is based on a global perception. In Geometry we operate with images like TV-images.

I do not understand why our students have troubles with Geometry: they are watching TV all the time. We just need to think how to use it. Anyway, images play more and more important role in modern life and so Geometry should play a bigger role in mathematics and in education. In physics it means that we should go back to geometrical intuition of Faraday (based on an adequate geometrical language) rather then to the Calculus used by Maxwell. People were impressed by Maxwell because he used Calculus: the most advanced language of his time.

Many talks in this conference (Dijkgraaf, Nekrasov, Schwarz, Seiberg, Vafa) are devoted to a search of proper geometrical language in physics. And never forget E. Cartan and always learn from Atiyah and Singer.

3.1. Exact language and Geometry.
(see the picture)

3.2. Matroids and Geometry.

I want to mention only one part of Geometry: Combinatorial Geometry and give you only two examples. One is a notion of matroids. I became interested in matroids when I understood that they give an adequate language for the geometry of hypergeometric functions by S. Gelfand, M. Graev, M. Kapranov, A. Zelevinsky, and me. With R. Macpherson I used matroids for a combinatorial description of cohomology classes of manifolds. Continuing this line Macpherson used oriented matroids for a description of combinatorial manifolds. We should also have a similar theory based on symplectic and Lagrangian matroids.

In particular, we should have a good "matroid" description for Chern–Simons classes.

3.3. Geometry and Protein Design.

Another example is my work with A. Kister "Combinatorics and geometrical structures of beta-proteins". Step by step, analyzing real structures, we are trying to create an adequate language for this subject. It is a new geometry for live objects.

4. FOURIER TRANSFORMS AND HYPERGEOMETRIC FUNCTIONS

In our search of an adequate language we should not be afraid to challenge the classics, even such classics as Euler. Quite recently we realized that our approach to hypergeometric functions can be based on the Fourier transform of double exponents like $e^{xe^{\sqrt{-1}\omega t}}$ where x and ω are complex and t is a real number. The Fourier transforms of such functions are functionals over analytic functions. For example, let $F(x,\omega,z)$ be the Fourier transform of the double exponent $e^{xe^{\sqrt{-1}\omega t}}$. Then

$$\langle F(x,\omega,z),\phi(z)\rangle = \sum_{k=0}^{\infty}\frac{x^k}{k!}\phi(-k\omega).$$

We may define the action of $F(x,\omega,z)$ as $\phi \mapsto \sum Res[f(z)\phi(z)]$ where $f(z)$ is a meromorphic function with simple poles in $k\omega$, $k=0,1,2,\ldots$.

The function $f(z)$ is defined up to addition of an analytic function. As a representative of this class we may choose the function

$$\Gamma_0(x,\omega,z) = \sum_{k=0}^{\infty}\frac{x^k}{k!}\frac{1}{z+k\omega},$$

or the function

$$(-x)^{-z/\omega}\Gamma(z/\omega).$$

We believe now that the function Γ_0 should replace the Euler function Γ in the theory of hypergeometric functions but this work with Graev and Retakh is in progress.

5. APPLIED MATHEMATICS, NON-LINEAR PDE AND BLOW-UP

My search for an adequate language is based in part on my work in applied mathematics. Sergey Novikov called me somewhere "an outstanding applied mathematician." I take it as a high compliment. I learned the importance of applied mathematics from Gauss. I think that the greatness of Gauss came in part because he had to deal with real-world problems like astronomy and so on and that Gauss admired computations. For example, I found recently that Gauss constructed the multiplication table for quaternions thirty years before Hamilton.

By the way, I remember my "mental conversation" with Gauss. When I discovered Fourier transforms of characters of abelian groups I had an idea that now I can make a revolution with Gauss sums and to change Number Theory. I even imagined telling this to Gauss. And then I realized that

Gauss, probably, would tell me: “You, young idiot! Don’t you think that I already knew it when I worked with my sums?”

5.1. PDE and Hironaka.

Working as an applied mathematician I realized the importance of the resolution of singularities while working with non-linear partial differential equations in late 1950s. I understood that we have to deal with a sequence of resolutions (blow-ups), by changing variables and adding new ones. So, I was fully prepared to embrace the great result of Hironaka. We studied his paper for a year. Hironaka’s theorem seems to have nothing to do with non-linear PDE. But for me it just shows the unity of mathematics.

Let me emphasize here that we still do not have a “Hironaka” theory for non-linear PDE.

5.2. Tricomi equation.

When the books by Bourbaki started to appear in Moscow, I asked ”In which volume a fundamental solution of the Tricomi equation will be published?”. Bourbaki did not publish this volume and it is time to do it myself.

The Tricomi equation is

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f.$$

It is elliptic for $y > 0$ and hyperbolic for $y < 0$. With J. Barros-Neto we found fundamental solutions for the Tricomi equation continuing works by Leray, Agmon and others.