Unity of Topological Field Theories

Cumrun Vafa

About 15 years ago Witten introduced three classes of quantum field theories in

2 , 3, 4 dimensions.

topological strings

$\Sigma \to M$

Chern-Simons gauge theory

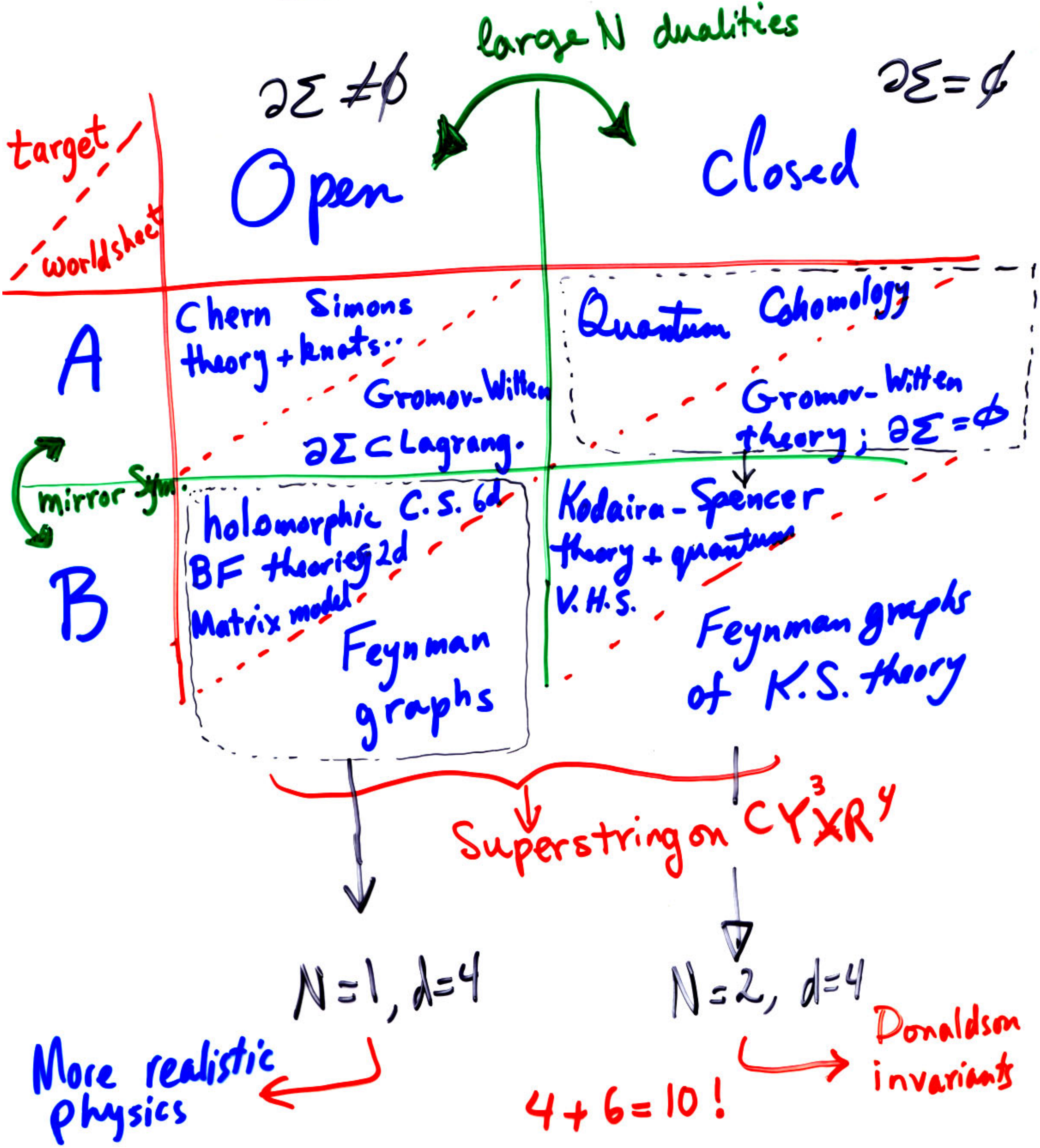
(knot invariants)

N=2 SYM (twisted)

Donaldson invariants

Now we know they are all deeply connected via strings.

Topological Strings



target worldsheet	$\partial\Sigma \neq \phi$ Open	$\partial\Sigma = \phi$ Closed
A	Chern Simons theory + knots.. Gromov-Witten $\partial\Sigma \subset$ Lagrang.	Quantum Cohomology Gromov-Witten theory; $\partial\Sigma = \phi$
B	holomorphic C.S. 6d BF theories 2d Matrix model Feynman graphs	Kodaira-Spencer theory + quantum V.H.S. Feynman graphs of K.S. theory

Superstring on $CY^3 \times R^4$

$N=1, d=4$

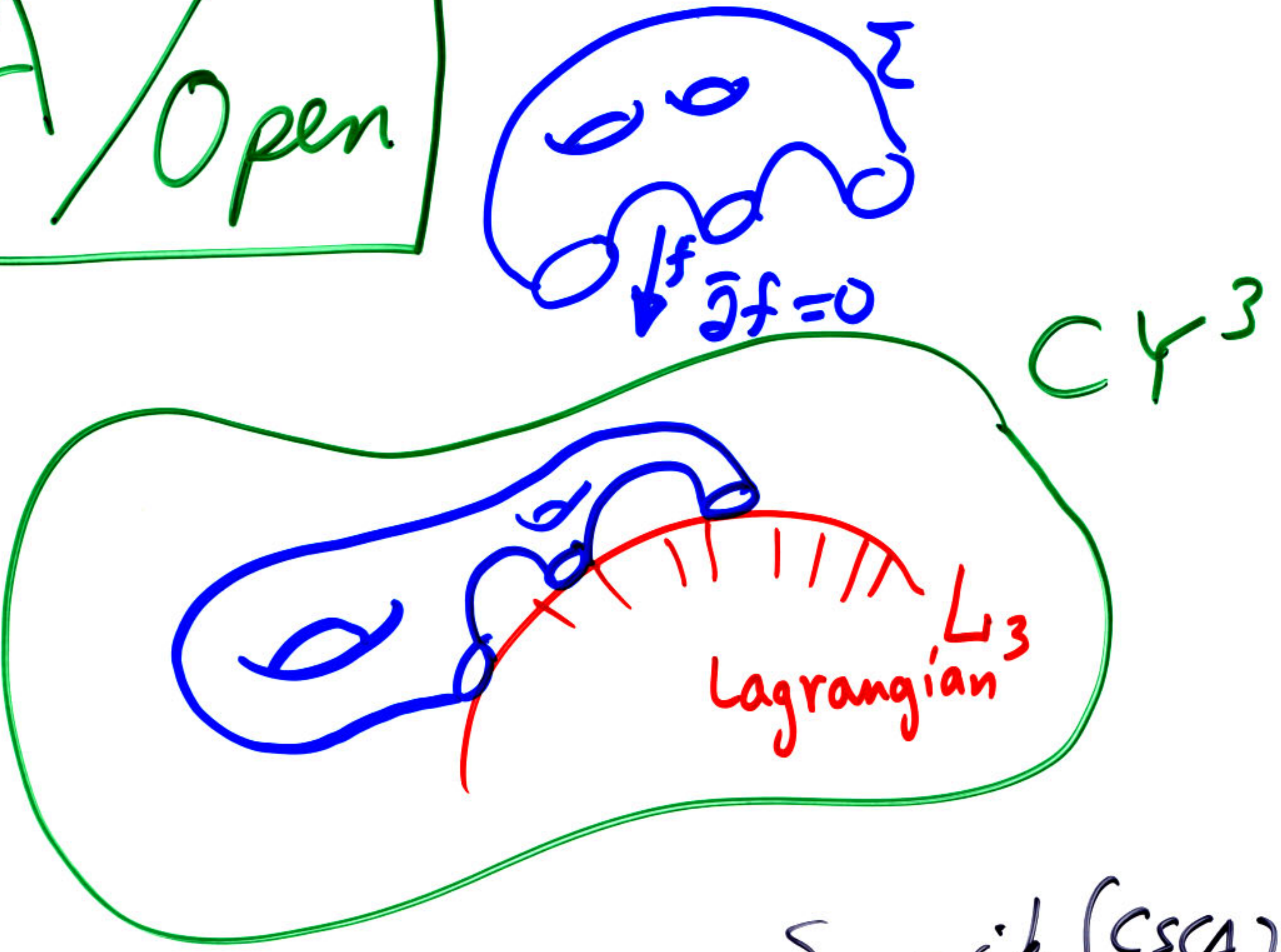
$N=2, d=4$

More realistic physics

$4+6=10$!

Donaldson invariants

A/Open

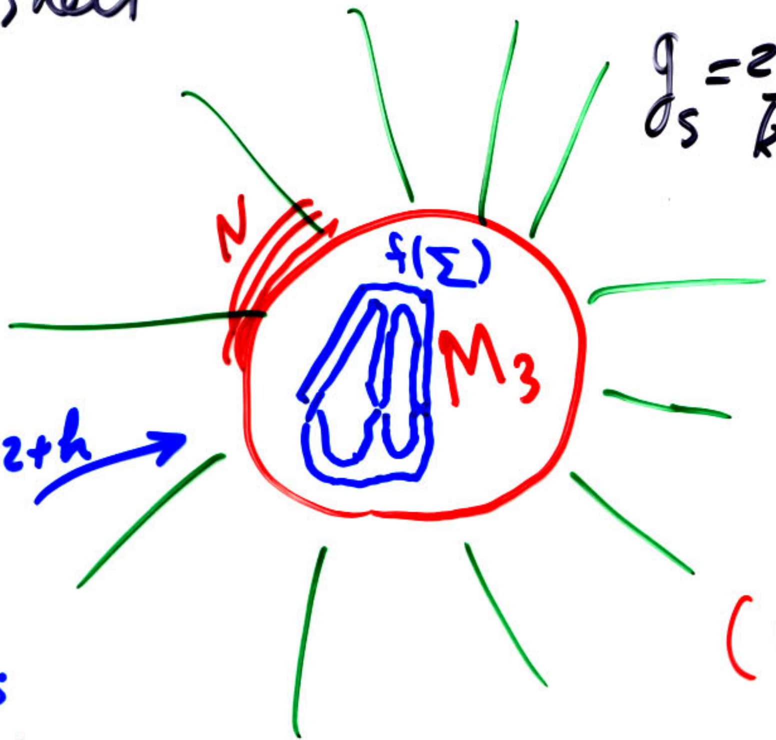


$S_{cs} = ik\int_{M^3} CS(A)$

target ↓

Worldsheet ↓

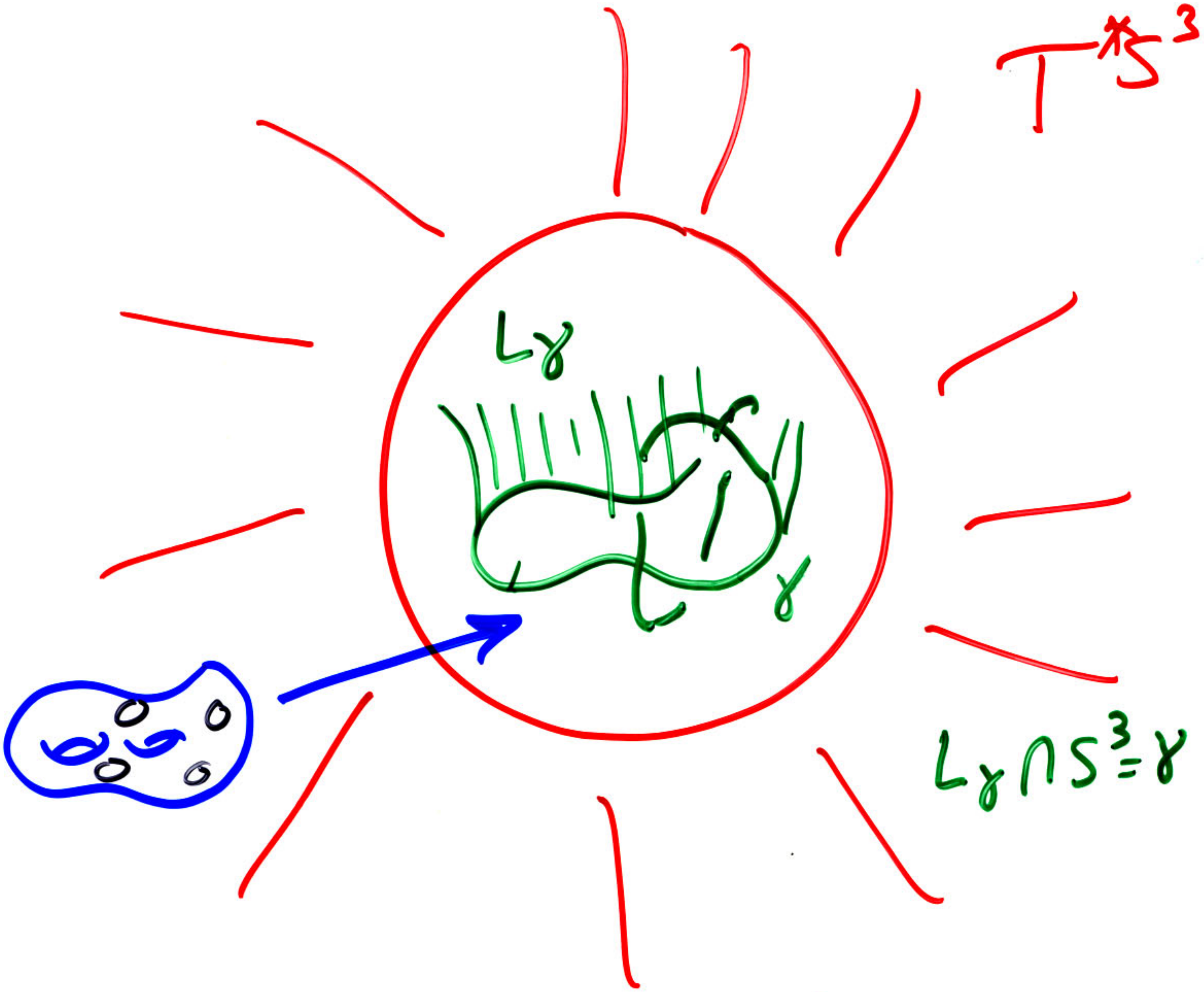
$g_s = \frac{2\pi i}{k+N}$



U(N) Chern-Simons on M^3.
(multiplicity N on M_3)

$N^h g_s^{2g-2+h}$ →

h: holes
g: genus



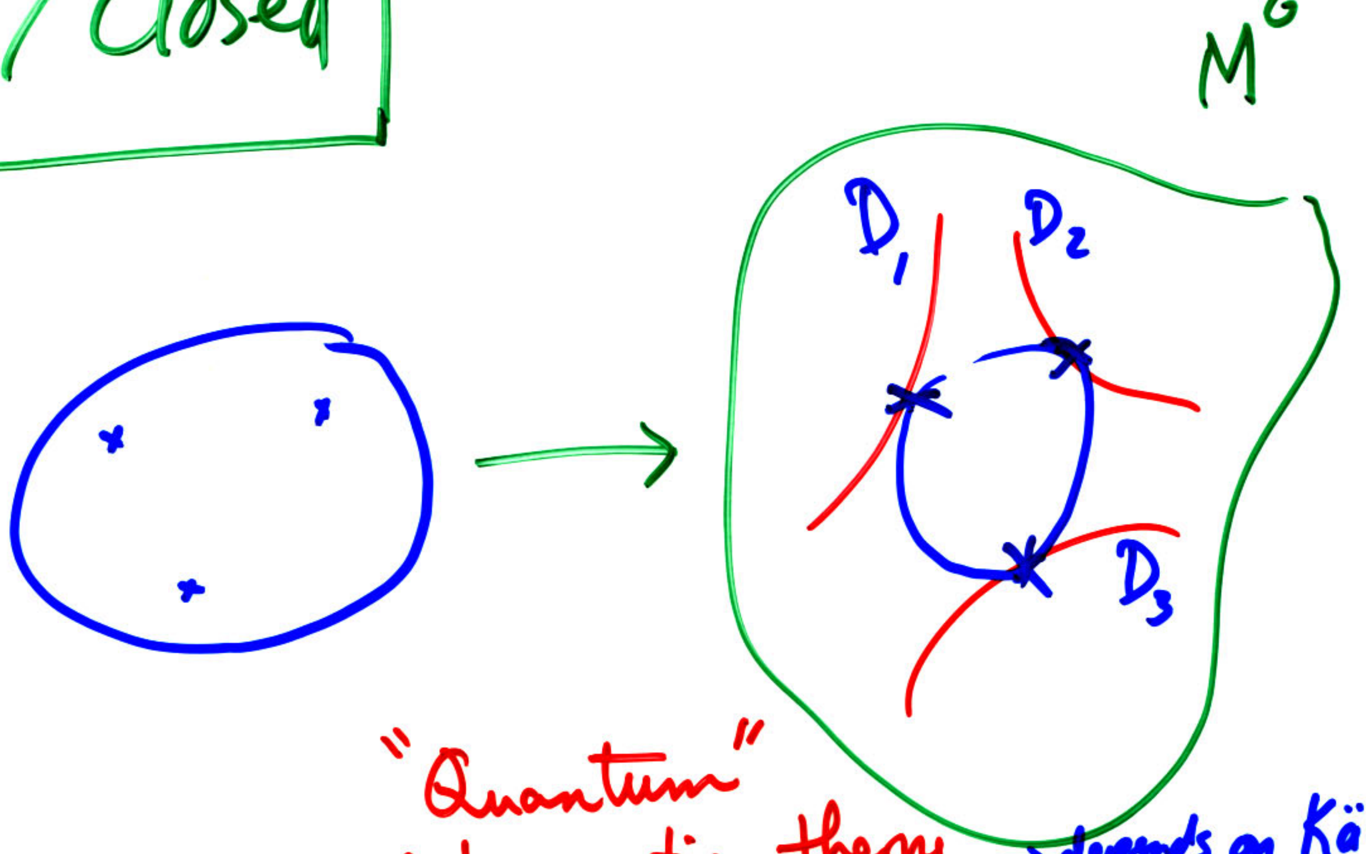
$L_\gamma \cap S^3 = \gamma$

$$\partial\Sigma \in [S^3]_N \cup [L_\gamma]_M$$

→ knot invariants on S^3.

$U(N)$, $g_s = \frac{2\pi i}{k+N}$; $M \leftrightarrow$ Rep. on γ. trace

A/closed



"Quantum"
intersection theory → depends on Kähler structure
+ higher genus corrections.

Interesting class of CY:

Non-compact toric 3-folds.

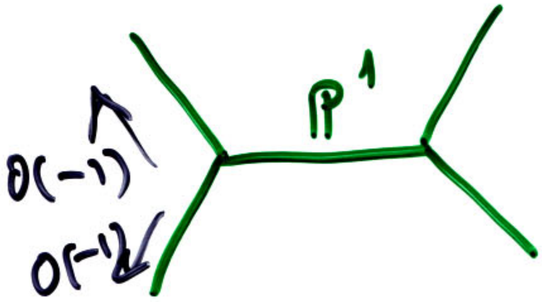
$$(\phi_1, \ldots\ldots, \phi_{n+3}) - \{loci\} \Big/ (C^*)^n$$

Examples:

$$(\phi_1, \phi_2, \phi_3, \phi_4)/C^*$$

weights $(1, 1, -1, -1)$

$$\begin{tikzcd} O(-1)+O(-1) \arrow[d] \\ \mathbb{P}^1 \end{tikzcd}$$

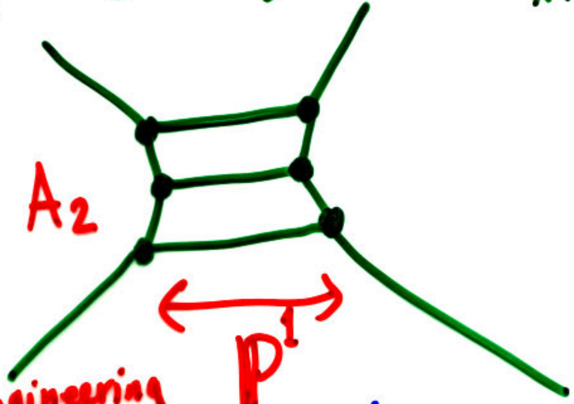


$$(\phi_1, \phi_2, \phi_3, \phi_4, \dots, \phi_{N+4})/(C^*)^{N+1}$$

$$\left.\begin{array}{l}(1 \quad -2 \quad 1 \quad 0 \quad \cdots\cdots \quad)\\ (0 \quad 1 \quad -2 \quad 1 \quad 0 \quad \cdots \quad)\\ (\qquad\qquad 1 \quad -2 \quad 1 \quad 0 \cdots \quad)\end{array}\right\} A_N$$

$$\left.\begin{array}{l}(\qquad\qquad\qquad\qquad \vdots \qquad \vdots \quad)\\ (\qquad \cdots \; 0 \quad -2 \quad \underbrace{1 \quad 1} \quad)\end{array}\right\} \mathbb{P}^1$$

resolved A_N singularity of $\mathbb{C}^2/\mathbb{Z}_{N+1}$ fibered over $\mathbb{P}^1$:



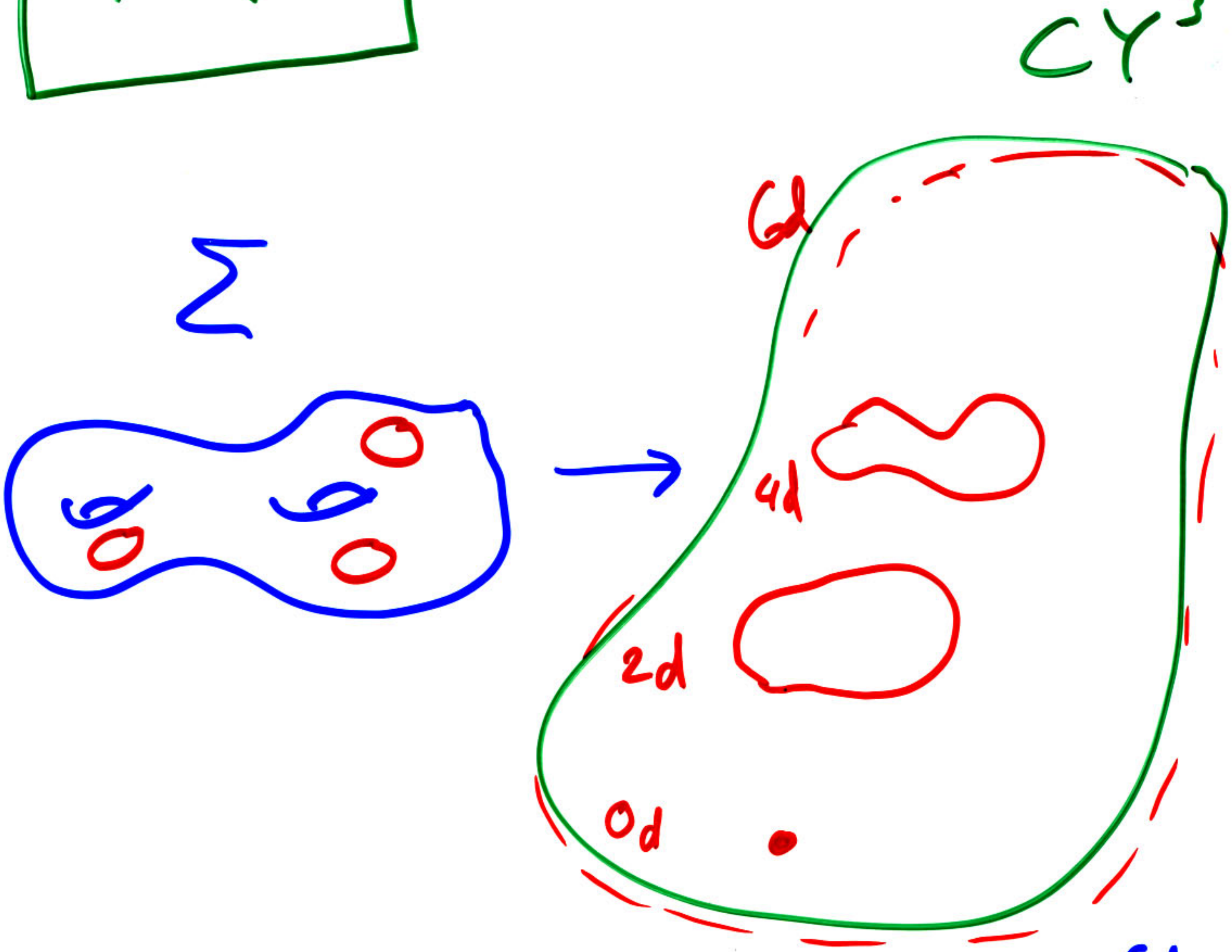
Superstring: $CY \times R^4$

$\Rightarrow$ $N=2$ $U(N)$ gauge theory on $\mathbb{R}^4$

rational curves $\xleftrightarrow{\text{Geometric Engineering}}$ $U(N)$ instantons on R^4

Klemm
Katz
V.

B/Open



$\partial\Sigma \subset$ holomorphic cycles $\begin{cases} 3 & 6d \\ 2 & 4d \\ 1 & 2d \\ 0 & 0d \end{cases}$

6d: $\partial\Sigma \subset CY^3$ (Neumann B.C.)

$\Rightarrow$ holomorphic Chern-Simons theory on CY^3:

$$S = \int_{CY^3} \Omega \wedge \mathrm{tr}\left(\bar{A}\,\bar{\partial}\bar{A} + \frac{2}{3}\bar{A}^3\right)$$

$\bar{A}$: hol. connection

Ω: holomorphic 3-form.

$\Big\downarrow$ 0d version

Matrix integrals

example: $S = \mathrm{tr}([X_i, X_j]X_k)\,\Omega^{ijk}$

$i,j,k = 1,2,3$

$+ \ldots$ deformations

Worldsheet diagrams →

Ribbon graphs of matrix model.

B/closed

$$A_i, B_i \in H_3(CY)$$

$$A_i \cap B_j = \delta_{ij}$$

$$\begin{cases} \int_{A_i} \Omega \sim t_i \\ \int_{B_i} \Omega \sim F_i \end{cases} \longrightarrow F_i = \partial_i F_0(t_i)$$

F_0 : genus 0- B model free energy:

Characterizes Variation of Hodge Structures on CY.

$$F = \sum_{g=0}^{\infty} F_g \lambda_S^{2g-2}$$

λ_S : string coupling constant.

"Quantum Kodaira-Spencer theory"

$F_1 \sim$ holomorphic Ray-Singer torsion.

$A \in \Omega^1(T)$

def of complex structure

$\bar{\partial}_A = \bar{\partial} + A$

$\bar{\partial}_A^2 = 0 \to \bar{\partial} A + [A, A] = 0$

Quantum Kodaira-Spencer theory has an action $S(A)$,

such that $\frac{\delta S}{\delta A} = 0 \to \bar{\partial} A + [A, A] = 0$

Quantum Kodaira-Spencer

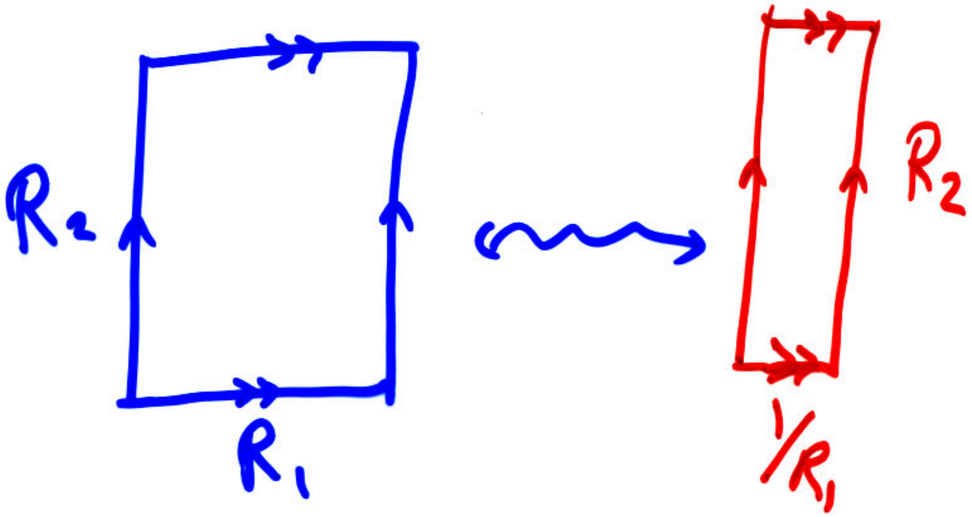
Mirror Symmetry

$$(CY_1)_A = (CY_2)_B$$

$$\underset{\overset{\uparrow}{\text{harder}}}{\text{Kahler}} \longleftrightarrow \underset{\overset{\uparrow}{\text{easier}}}{\text{Complex}}$$

Basic example:

T^2:



$$A = R_1 R_2 = -i\tau = R_2 R_1$$

Mirror for local CY (toric):

Hori, V.

$$(\phi_1, \dots, \phi_{N+3}) / C^{*N}$$

$$(q^i_1, \dots, q^i_{N+3}) \leftarrow \text{weights} \quad i = 1, \dots, N$$

$$\updownarrow$$

$$(Y_1, \dots, Y_{N+3})$$

mirror variables

$$\mathrm{Re}[Y_i] = |\phi_i|^2, \quad (\mathrm{Im}\, Y_i) \xleftrightarrow[R \to 1/R]{} \arg(\phi_i)$$

$$\sum_{j=1}^{N+3} q^i_j Y^j = t_i$$

$$y_i \equiv e^{-Y_i}$$

$$F(e^u, e^v) \to$$

Riemann surface $F = 0$

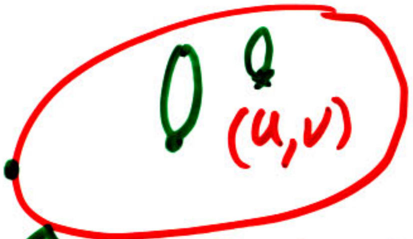
mirror CY:

$$\begin{cases} XZ - \overbrace{\sum_{i=1}^{N+3} \left(\frac{y_i}{y_1}\right)} = 0 \\ \prod_j y_j^{q^i_j} = e^{-t_i} \end{cases} \qquad C \times C \times C^{*2}$$

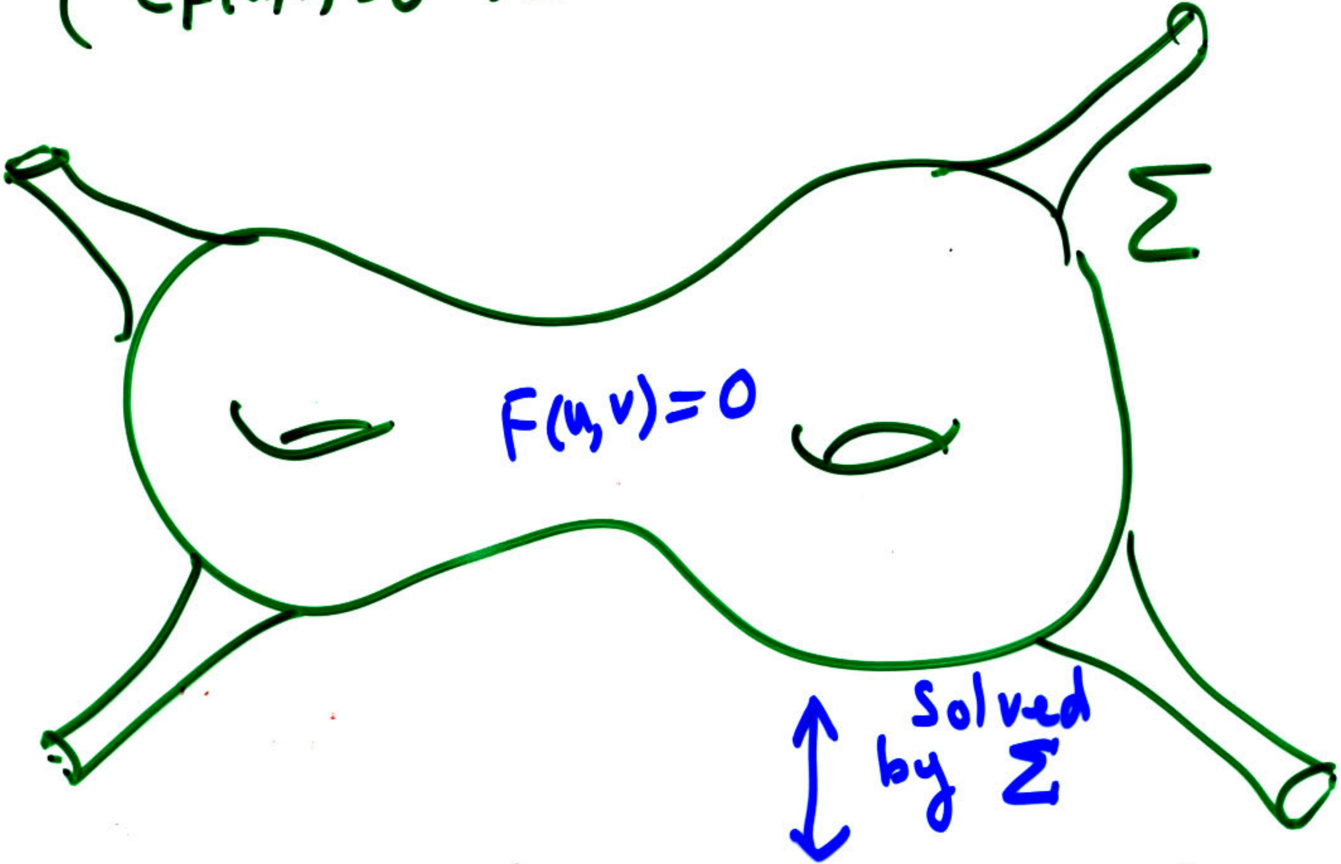
$$\text{V.H.S.} \rightarrow xz - F(e^u, e^v) = 0$$

$$\int \Omega = \int \frac{dx}{x}\, du\, dv = \int_{\text{Domain}} du\, dv = \int_{H_1(\Sigma)} u\, dv$$

$$\partial D \equiv (F = 0)$$

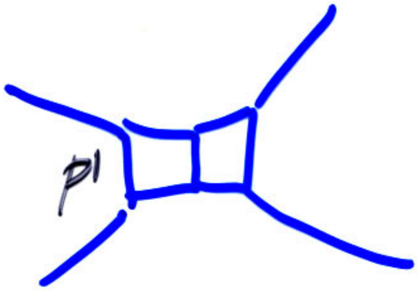


(↑ $F(u,v) = 0$: Σ)



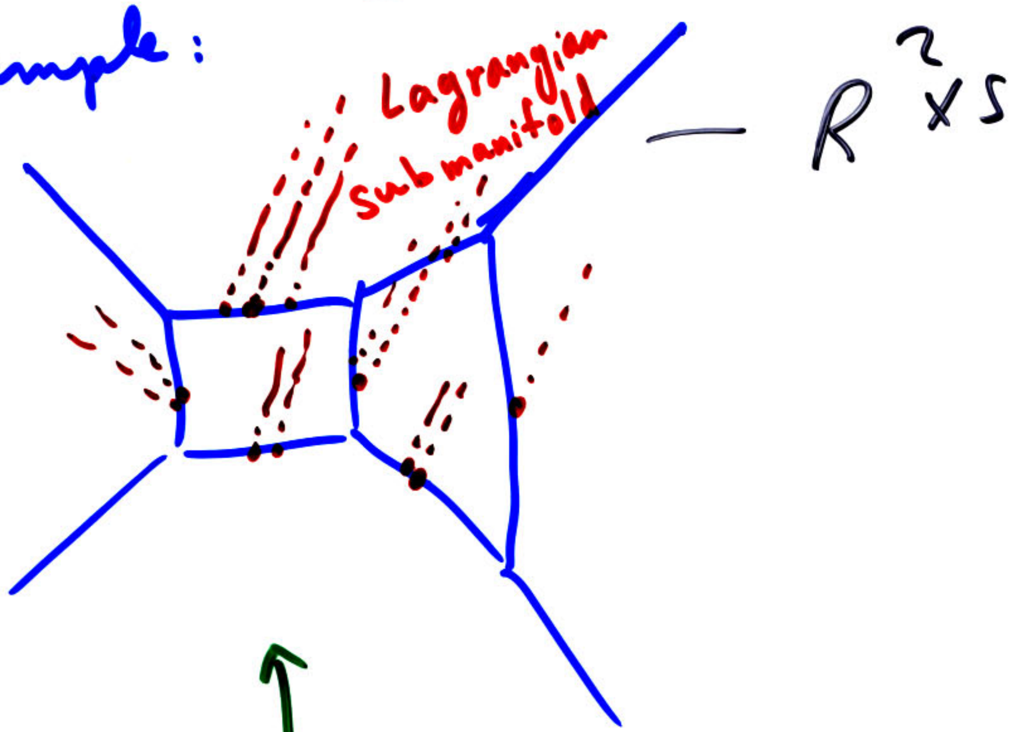
↕ Solved by Σ

A_2 gauge theory

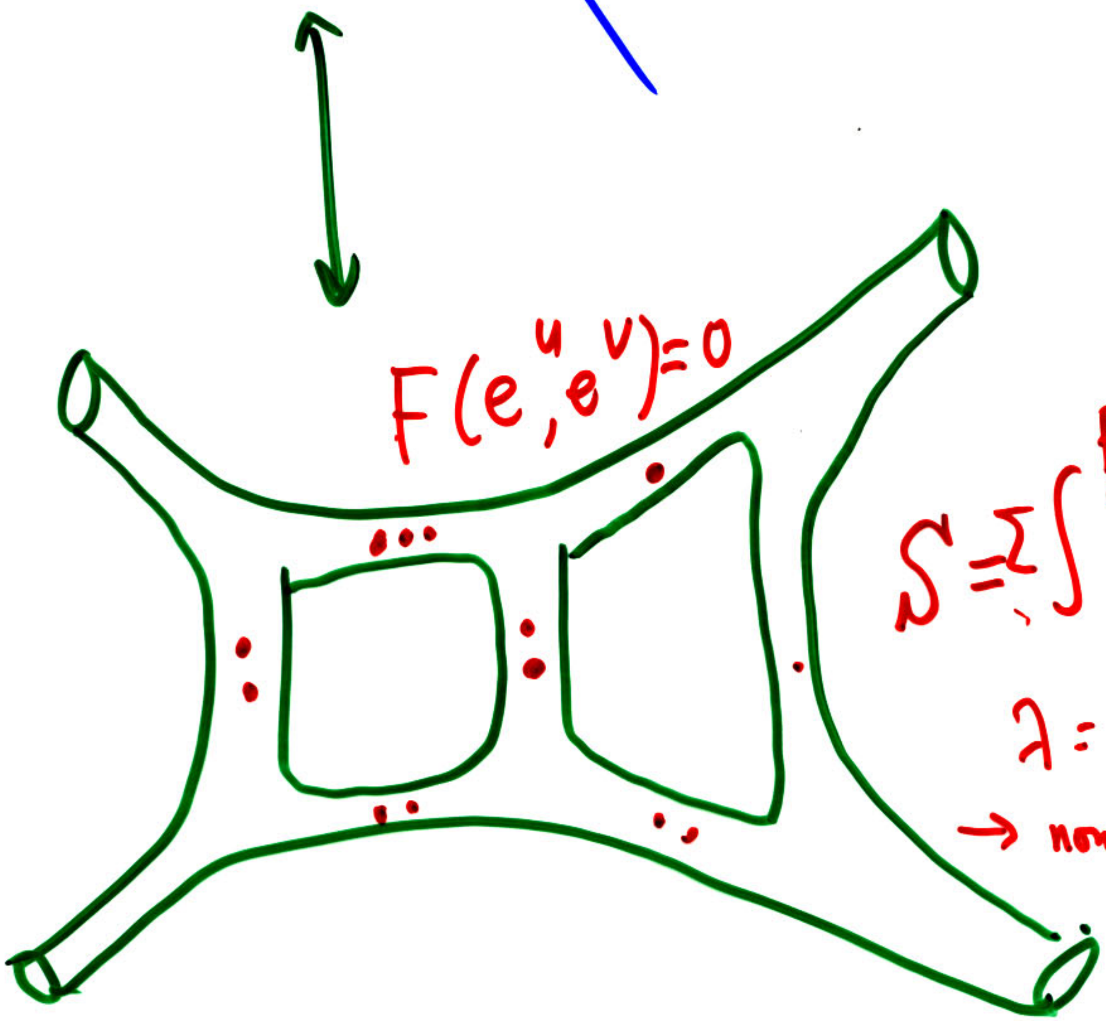


Mirror Symmetry
With Branes ($\partial\Sigma \neq \emptyset$)

example:



Aganagic V.

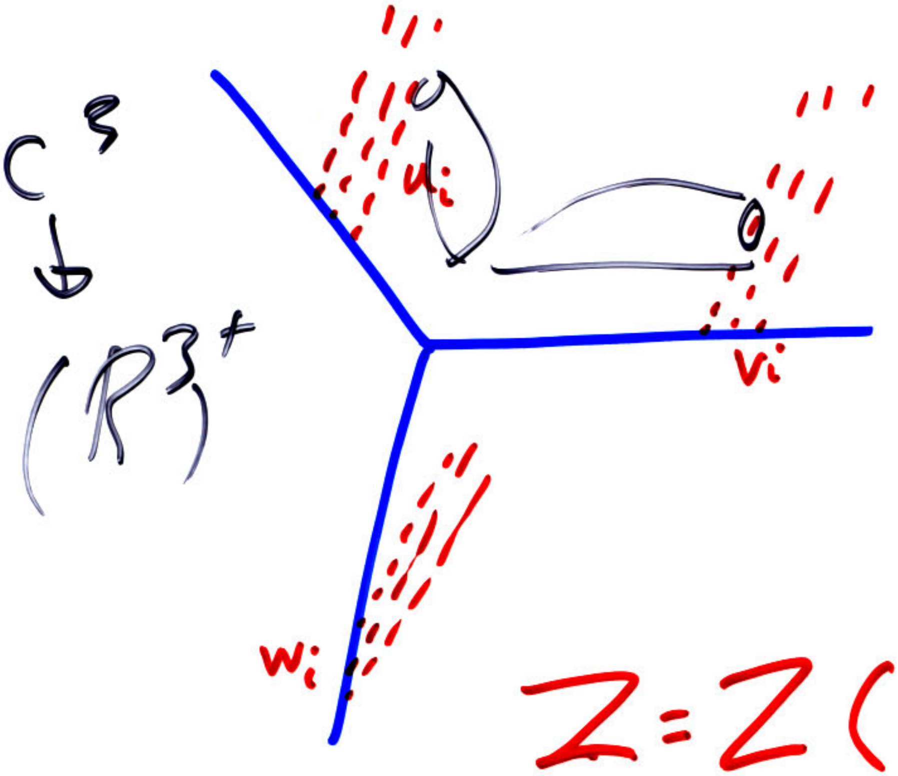


$S = \sum_i \int^{p_i} \lambda$ classica

$\lambda = u\,dv$

→ non-trivial g_s corrections

All A-model amplitudes on local toric CY can be reduced to

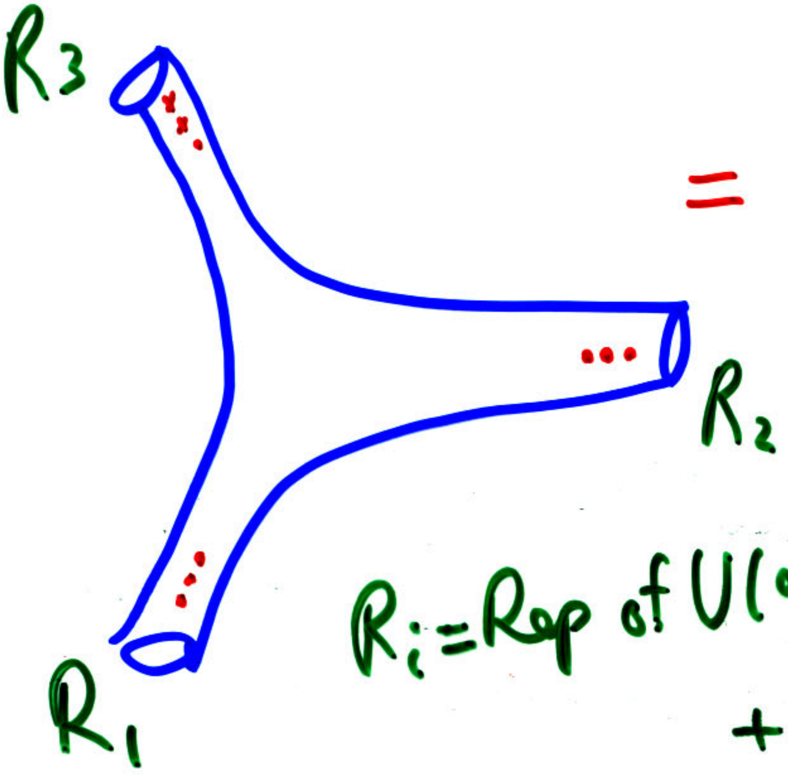


$$U_0 = \begin{pmatrix} e^{u_1} & & \\ & \ddots & \\ & & e^{u_N} \end{pmatrix}$$

$$V_0 = \begin{pmatrix} e^{v_1} & & \\ & \ddots & \\ & & e^{v_n} \end{pmatrix}$$

$$W_0 = \begin{pmatrix} e^{w_1} & & \\ & \ddots & \\ & & e^{w_n} \end{pmatrix}$$

$$Z = Z(U_0, V_0, W_0)$$



$$= \mathrm{tr}_{R_1} U_0 \; \mathrm{tr}_{R_2} V_0 \; \mathrm{tr}_{R_3} W_0 [C_{R_1 R_2 R_3}]$$

has been computed recently

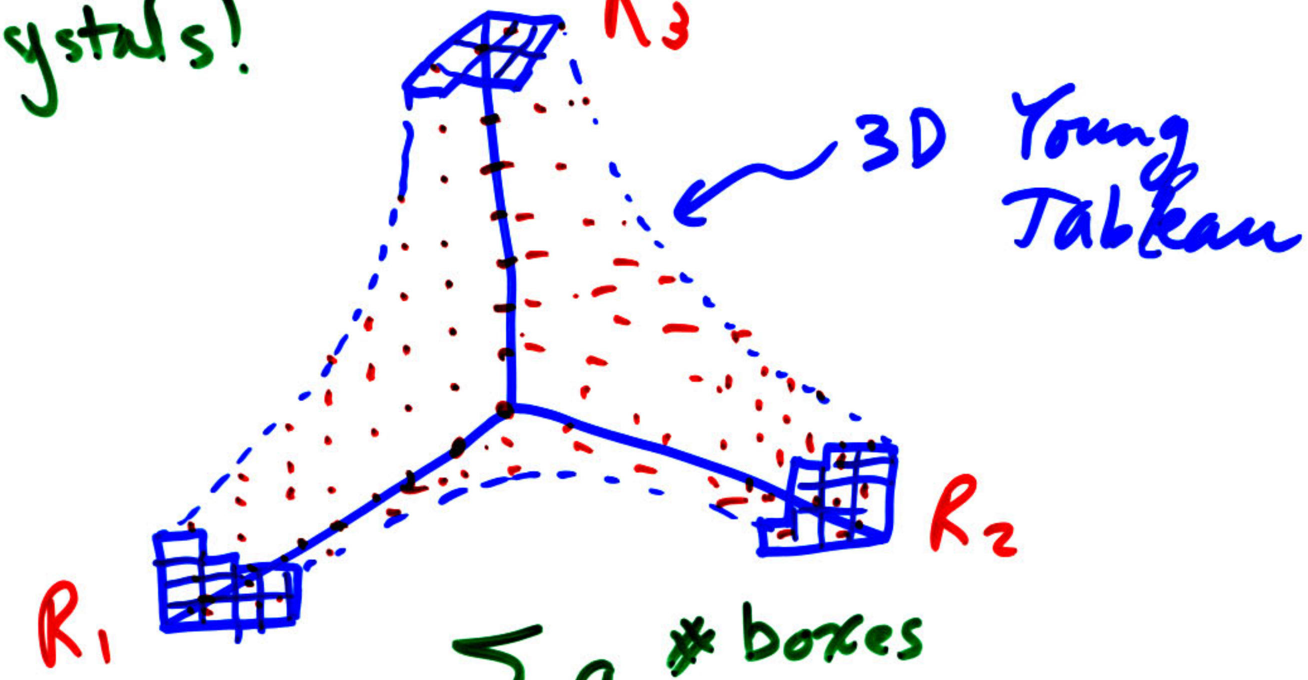
R_i = Rep of $U(\infty)$

Iqbal, Kashani-Poor

\+ Aganagic, Klemm, Marino, V.
// , Dijkgraaf

$C_{R_1 R_2 R_3}(q)$ $\qquad$ $q = e^{-g_s}$

There seems to be a new deep relation to classical 3D crystals!



$$C_{R_1 R_2 R_3} \propto \sum_{\substack{\text{3D} \\ \text{Young Tableau} \\ \text{which asymptote } \{R_1, R_2, R_3\}}} q^{\#\text{ boxes}}$$

Okounkov, Reshetikhin, V.

$g_s >> 1 \rightarrow$ (Calabi Yau $\longrightarrow$ 3d lattice)

A particular example of this:

$$Z = \sum_{\substack{\text{3d Young} \\ \text{tableau}}} q^{\#\text{boxes}} = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)^n}$$

with no fixed asymptotic restrictions

McMahon's formula

$$\Big\Vert$$

$$\Big\downarrow \text{A-model}$$

$$Z = \exp\left(\sum_g \left[\int_{\bar{\mathcal{M}}_g} c_{g-1}^3(H)\right] \lambda_s^{2g-2}\right)$$

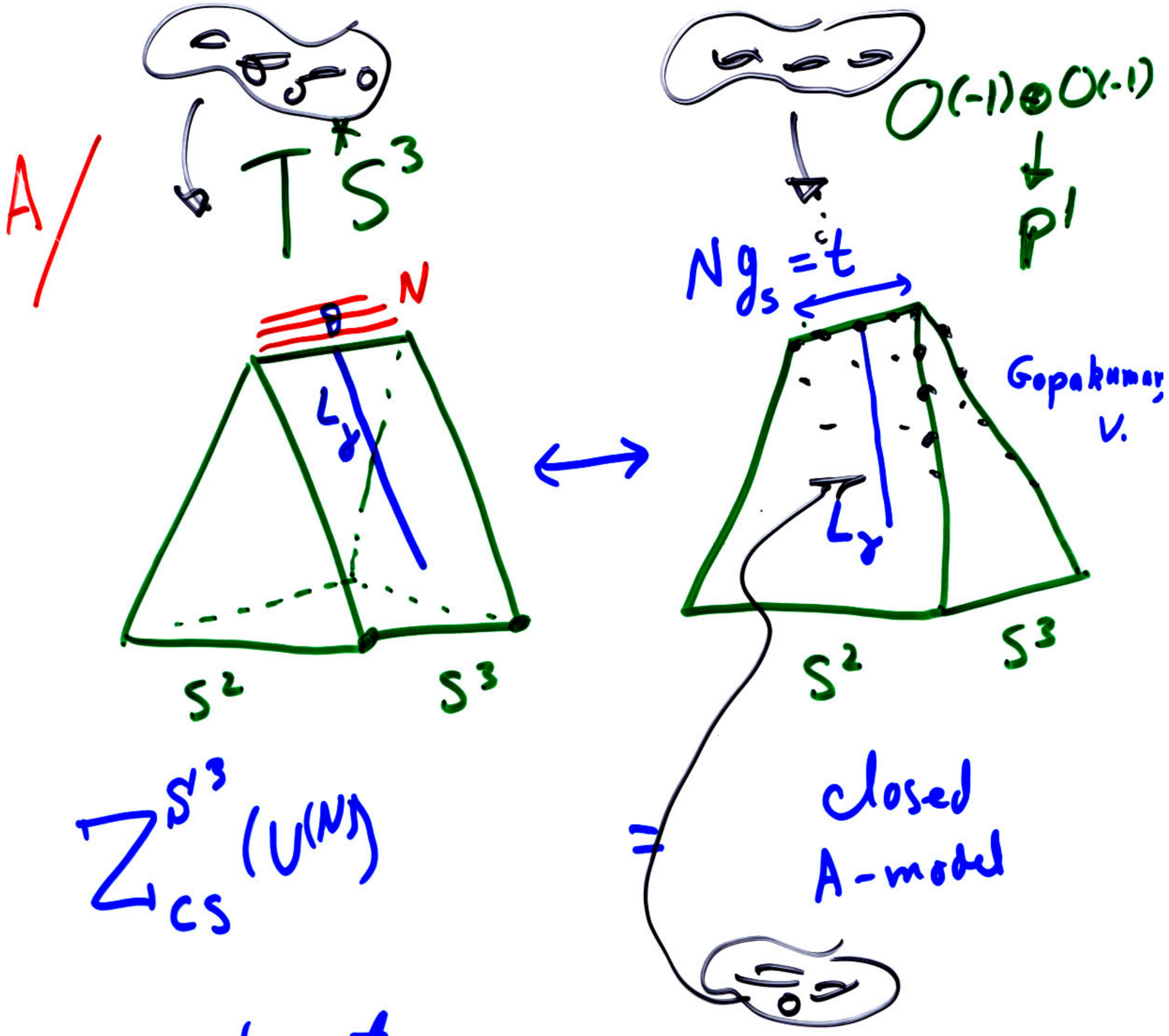
$$e^{-\lambda_s} = q$$

Gopakumar, V.

Faber, Pandharipande

Large N dualities

Open ⟷ closed



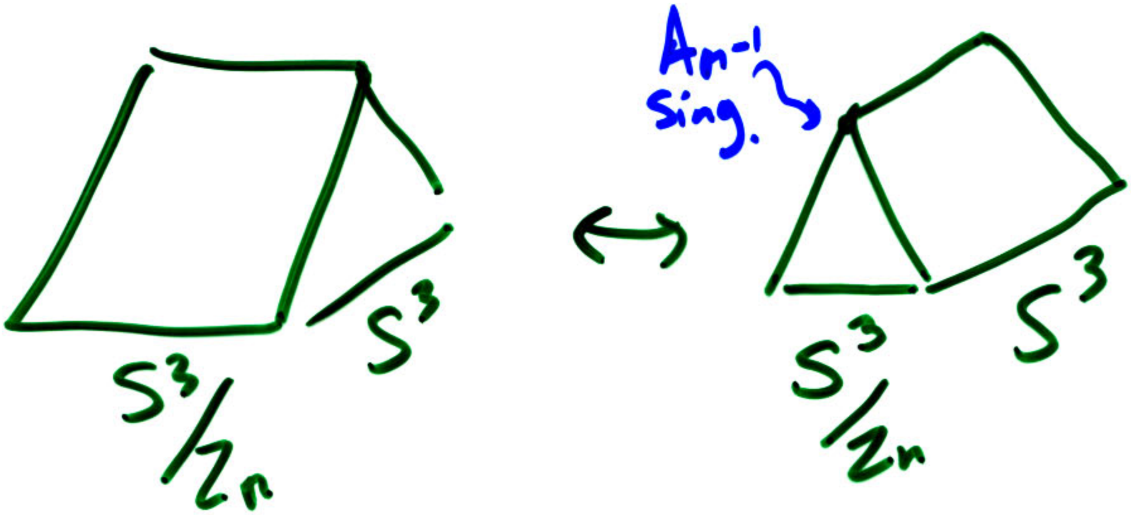
$Z_{CS}^{S^3}(U(N))$ = closed A-model

+ knots = open A-model

This duality can be lifted to a geometric statement for M-theory on G_2 manifolds:

$$\text{Cone over}\left(\frac{S^3}{Z_n}\times S^3\right)\rightarrow \text{Cone Over}\,(S^3\times S^3/Z_n)$$

Atiyah, Maldacena, V.



B-model

Open

Closed

matrix integral

$$\int D\Phi \, e^{-\frac{W(\phi)}{g_s}} \longleftrightarrow xy + \omega^2 + \omega W'(z) + f_{n-1}(z) = 0$$

CY in $\mathbb{C}^4$

(x, y, ω, z)

$$W(\phi) = \mathrm{tr} \sum_{r=0}^{n+1} \frac{\phi^r}{r} \alpha_r$$

Dijkgraaf, V.

$$\omega^2 + \omega W'(z) + f_{n-1}(z) = 0$$

$$\omega = \left\langle \mathrm{tr} \frac{1}{z - \phi} \right\rangle$$

$$\Phi = \begin{pmatrix} \overbrace{a_1 \ddots}^{N_1} & & \\ & a_2 \ddots & \\ & & a_n \ddots \end{pmatrix}$$

leading large N behavior = planar graphs

(standard techniques)

$$\frac{1}{g_s} \int_{A_i} \omega \, dz = N_i$$

Connections to $N=1$ SYM

$d=4$

Dijkgraaf, V.

As mentioned

B-model open version in 0-d:

$$\int D\phi_i \, e^{W(\phi_i)} \qquad \text{matrix model}$$

$$\Big\updownarrow \ \text{embed} : \underbrace{R}^4 \times CY^3$$

$N=1$ SYM , $d=4$
with ϕ_i matter, $W(\phi_i)$
superpotential.

$W'(\phi_i) = 0 \rightarrow$

certain vacua

$(M_1, \ldots, M_n)$

$\rightarrow$ Planar graphs $(M_1, \ldots, M_n)_{g_s}$

$|?$ $F_0(S_1, \ldots, S_n)$

$S_i = M_i g_s$

Bershadsky et. al.

$$W(S_i) = N_i \frac{\overbrace{\partial F_0}^{F_i}}{\partial S_i} - \tau \sum S_i$$

spacetime superpotential

S_i = "glueball fields"

$W'(S_i) = 0 \Rightarrow$ Vacuum geometry!

including instantons!

This leads to a Perturbative window into Non-perturbative (i.e. instanton) physics.

Example:

N=1 theory with 3-adjoints X, Y, Z with $W(X,Y,Z) = \mathrm{tr}[X,Y]Z$

$\cong$ (N=4 Yang-Mills)

$$\int DX\, DY\, DZ\; e^{-\frac{\mathrm{tr}}{g_s}[X,Y]Z + m(X^2+Y^2+Z^2)}$$

N=4 coupling constant

$$\Rightarrow W(S) = N\frac{\partial F}{\partial S} - \tau S$$

planar graphs

$\hat{\tau} = \tau/N$

$$\frac{\partial W}{\partial S} = 0 \Rightarrow W\Big|_{min} = m^3 E_2(\hat{\tau})$$

$$E_2(\hat{\tau}) \approx \sum \sigma_1(n) q^n$$

Montonen-Olive duality $\hat{\tau} \to -1/\hat{\tau}$ $\mathcal{N}=4$ YM

↕

Modularity of $E_2(\hat{\tau})$

Topological strings → strong coupling dualities in gauge theories.

A Highly non-trivial

but satisfying picture!

Unity of Topological

Field theories !