"Despite having a classification of the building blocks of all finite groups, we are a long way from understanding the chemicals we can build from these atoms. The challenge begins with trying to understand the groups built out of the simplest of the simple groups, a cyclic group of order p. Trying to classify groups of prime power order has long been considered a difficult and wild problem. A major theme throughout the last two hundred and fifty years of mathematics is the power of an analytic function called a zeta function to bridge that divide between dangerous uncertainty and the quest for order."

- Marcus du Sautoy, "Zeta functions of groups: The quest for order versus the flight from ennui."

The Trivial Notions Seminar

Proudly Announces

Zeta Functions of Groups and Rings

A talk by

Nathan Kaplan

Abstract

Let G be an infinite group and $a_n(G)$ be the number of subgroups of G of index n. The zeta function of G is

$$\zeta_G(s) = \sum_{n=1}^{\infty} a_n(G) n^{-s} = \sum_{H \leq G} [G : H]^{-s}.$$

For $G = \mathbb{Z}$ this is the classical Riemann zeta function.

These zeta functions of groups share many of the key properties that make classical zeta functions so interesting. These functions usually have Euler product expansions with local factors given by p-adic integrals. We will show how to compute some examples of these integrals and prove that

$$\zeta_{\mathbb{Z}^n}(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-(n-1)).$$

We will also discuss questions about rationality, meromorphic continuation, and functional equations.

Similar zeta functions can be defined that count other types of arithmetic objects. For example, the Dedekind zeta function of a number field can be thought of as a zeta function for ideals. We will sketch a proof that the zeta function for subrings of $\mathbb{Z}^3$ is given by

$$\frac{\zeta(s)^3\zeta(3s-1)}{\zeta(2s)^2}.$$

We will also consider zeta functions that count orders in a given number field.

Thursday May 2^{nd}, at <u>1:30</u> pm
Science Center 310